Musical Temperament

Daniel Adam Steck
Oregon Center for Optics and Department of Physics, University of Oregon

Musical Temperament

Daniel Adam Steck

Oregon Center for Optics and Department of Physics, University of Oregon

Original revision posted 25 January 2015.

This is revision 0.2.3, 24 November 2015.

Cite this document as:
Daniel A. Steck, *Musical Temperament*, available online at `http://steck.us/teaching` (revision 0.2.3, 24 November 2015).

Author contact information:
Daniel Steck
Department of Physics
1274 University of Oregon
Eugene, Oregon 97403-1274
dsteck@uoregon.edu

Acknowledgements

Special thanks to Natascha Reich (U. Oregon), for

- *Many* helpful comments, corrections, suggestions, and criticisms.
- Asking interesting questions.
- Help in understanding old German.
- Help in selecting the musical examples for Chapter 6, and advice in what to listen for in various temperaments.

For comments and corrections, thanks also to:

- Raven Perry (U. Oregon)

Contents

Using This Book

Assumptions and Background

Musical temperament is a subject both musical and mathematical. Some basic skill in both areas will help in going through this book. Chapter 1 reviews musical basics, including scales, intervals, and pitch notation; it's *crucial* to understand these before going on to the rest of the material.

Some basic aspects of sound are assumed, such as what is a sound wave; what is frequency; what are harmonics and partials.

Some basic level of skill in mathematics is also necessary, but nothing *too* horrific: arithmetic, ratios, logarithms, and some basic algebra will get you where you need to go. The first few chapters are lighter on math, and the difficulty gradually increases. Chapter 8 (the continued-fraction basis of scales) requires some more comfort with math than the other chapters.

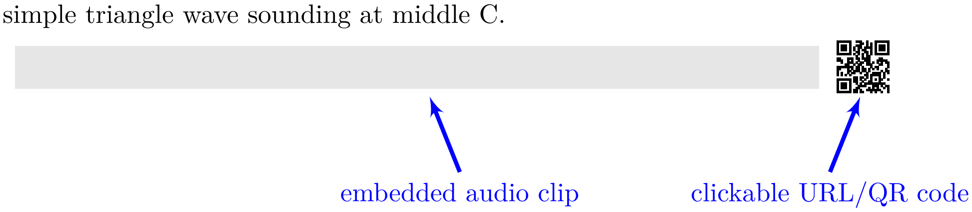
Most of the material here is designed for a college-leve, general-education course in the physics or mathematics of music.

Exercises

Correspondingly, most of the exercises involve some mathematics, but the earlier ones in each chapter are simpler than the later ones. More challenging problems are marked with "**(!)**" after the problem number; some of these may require a fair amount of mathematical sophistication. The ones not marked are good exercises for just about anyone.

Audio Examples

Audio examples are critical in any discussion of musical temperament. Throughout this book, you will see audio-clip references like the one below (there are 169 audio clips in this book). As an example, the audio clip below is a simple triangle wave sounding at middle C.



What you see here is two graphical objects. On the left is the better option, *in principle*, the embedded sound clip. Unfortunately, it may also be the *complicated* option. You can try clicking on it now to see if it will work. If it does, you're golden, and you can skip to the next section. If not, keep reading.

On the right, you see a complex-looking square. This provides *two* ways to access the sound clip, both via the web.

- First, this acts as a **clickable URL** from within the pdf document. Click on it, and your pdf reader should automatically switch to a web browser, which will load an online copy of the audio clip in mp3 format.

- Second, this is a **QR code**, which you can scan with a smart phone or tablet. This will get you the same URL as the click. This option is handy if you've printed out the book, because you can use your mobile device to read the URL off the paper and play the sound sample. You can also scan the code off your computer screen, if it's zoomed in enough, and if for some reason this is easier than just clicking on the code. Note that the QR codes are fairly small (printed at 7 mm on paper) so they aren't too obtrusive in the document. This means, however, that not all devices and reader apps will be happy with these codes—a high-resolution camera and decent app are necessary. However the **i-nigma** app on both an iPhone 5S and Samsung Galaxy S III worked great in testing. Another free app on the same iPhone would only scan larger codes; try out different ones until it works.

Now back to the embedded audio clip on the left.

Why is the state of embedded multimedia in PDF documents so terrible? In part, it is Adobe's habit of releasing software that leaves a *lot* to be desired, in terms of both security and aesthetics. But also, few developers of alternative PDF viewers have provided better options. It is as if PDF multimedia is stuck in 1990's limbo—PDF multimedia hasn't improved much since then, but it *has* gottem more complicated.

This document is generated using LaTeX; I include some details of this process in the description below in case it is useful to counter any problems that may crop up. A good summary of the status of multimedia in PDFs, particularly as generated by LaTeX, as of late 2013 is given on a French-language site.[1,2] The main option for viewing these files with embedded sound clips functioning is Adobe Acrobat (or the free Acrobat Reader[3]), but sadly, this software is bloated, slow, and renders text poorly. So we will review some of the options for viewing these files.

Specifically, this document is created in LaTeX with the `media9.sty` package.[4] The mp3 sound clips are embedded in the pdf document using Rich Media Annotations, along with a self-contained Flash player (`APlayer.swf`) to handle playback. Options for using these clips:

- Adobe Acrobat/Reader (version 10 and higher, on OS X) works reasonably well in playing back the clips. Controls are supposed to be available, but do not seem to work beyond play and stop.
- On iOS and Android devices, PDF Expert (version 5)[5] handles the sound clips well and renders text nicely. Highly recommended.
- Things that seem like they should work, but don't: Preview.app on OS X, ezPDF in iOS.

If you have success (or not) in other readers or operating systems, please let me know.

Other Hyperlinks and Navigating this Document

To make it easier to access information within and beyond this document, there are many hyperlinks throughout. To keep the document "pretty," the hyperlinks are not highlighted in colors or boxes by default. Some of the more obvious ones are the spelled-out URLs, like `http://steck.us`, are clickable, as are the QR codes mentioned above. Some of the less obvious ones are:

- DOI (document object identifier) codes in article citations (for locating articles online).
- ISBN codes in book citations (these resolve to pages on `amazon.com`).
- The "*op. cit.*" abbreviation in repeated footnote references (will take you to the original footnote reference).
- Section titles in the Contents.

[1] `http://linuxfr.org/users/gouttegd/journaux/integrer-des-videos-dans-des-fichiers-pdf`
[2] English translation via Google translate: `http://tinyurl.com/pzklmza`.
[3] `http://get.adobe.com/reader/`
[4] `http://www.ctan.org/tex-archive/macros/latex/contrib/media9`
[5] `https://readdle.com/products/pdfexpert5`

- Page numbers in the Index.
- Chapter, section, page, and equation numbers throughout the text.

Introduction

It is reasonably common knowledge that people tend to play music using a certain set of musical pitches—or a musical **scale**. Scales are especially helpful when several musicians play together, or even when a single musician plays several notes together. With the common musical "language" of scales, the combination of many pitches in a song can be sublime; without this language, the result is usually chaos.

You might think that this is pretty straightforward. Musical instruments like pianos, guitars, and flutes all come with notes from the same scale, right? And if you want to tune your instrument, you just whip out an electronic tuner, which tells you if the frequency is set correctly, right?

Well, as it turns out, not so much. The business of setting up musical scales is tricky and even controversial. The *math* involved isn't difficult—for the most part, it's fairly straightforward arithmetic. (And yes, math *is* involved in music.) However, it turns out that in putting together scales, you always have to make *compromises*. Specifically, the compromises come in optimizing the tuning of harmonies. It turns out that not all combinations of pitches can be perfectly "in tune" in a single scale, even when it seems like they *should* be—that's where the trickiness comes in. (We'll define more precisely what we mean by all this later.) Thus comes the art of **musical temperament**, or the art of deciding *which* combinations to favor over others, or maybe to be democratic and treat them all in the same way.

Musical temperament is an interesting subject because it unites the precision of mathematics and physics with the art of music. In one sense, musical temperament *describes* the practice of musicians and how they play music that sounds nice. On the other hand, temperament acknowledges some basic mathematical constraints on how pitches can fit together. It even *prescribes* some of the performance of music, because it helps musicians make some decisions before performing music, in how their instrument should be tuned to suit the piece at hand.

In particular, it is combination of precision and the "human factor" that makes temperament challenging and interesting, and leads to experts raging at each other over the centuries. To illustrate some of the human factor before going on to the meat of things, let's look at an example that is very interesting, though probably not particularly important to music. The **frequency** f of a sound is a measurable, physical quantity: assuming a sound is **periodic** (i.e., it repeats itself after a time T, called the **period**), the frequency is the *rate* at which it repeats, or $f = 1/T$. Frequency is measured in Hertz, which is just another way of saying "repetitions per second." (An additional complication is that most musical sounds are not *quite* periodic, but most are close enough that we won't worry about this for now.) The **pitch** of a sound is an expression of how we *perceive* the sound, and may even describe the musical *function* of a sound in the context of a song. The pitch then has the human element. Higher frequencies generally correspond to higher pitches, and lower frequencies to lower pitches. However, the correspondence is not exact. Listen to the sound clips below. All the clips alternate between two tones several times, one loud and one soft—the louder one has 20 times the **amplitude** of the softer one (amplitude here meaning the maximum displacement of the speaker when playing back the sound waves). The first two clips are in the low-frequency range, the second two in a higher-frequency range, and the last two are yet higher in frequency. The question is, in each case, is the louder tone slightly *higher or lower in frequency* than the softer one? Or are they the same?

We'll explain what is going on afterwards, but **listen to these first** before going on! Try to use a decent sound system (headphones or speakers that don't distort much), and adjust the volume so you can comfortably hear both the loud and soft tones.

In most or all of the examples, you probably heard slight differences in pitch between the tone. Sometimes, the louder tone seems higher in pitch, sometimes lower. What may be surprising is that the louder and softer tones **have the same frequency** within each sound clip. That is, any variation in pitch within each sound clip is *purely* due to human perception. Thus pitch depends not only on the frequency of sound, but also on intensity.[6]

Here is a bit more detail about the clips, and how the pitches varies (at least, according to *my* ears).

1. 200-Hz sine wave. The louder tone has the lower pitch, and the effect is stronger at higher overall volume.

2. 200-Hz asymmetric triangle wave. At lower overall volume, the louder tone has a slightly lower pitch; at higher overall volume, the louder tone has a higher pitch.

3. 2-kHz sine wave. At lower overall volume, the louder tone has a higher pitch; at higher overall volume, the louder tone has a lower pitch.

4. 2-kHz asymmetric triangle wave. The louder tone has a higher pitch at higher or lower overall volume.

5. 5-kHz sine wave. The louder tone has a higher pitch.

6. 5-kHz asymmetric triangle wave. The louder tone has a higher pitch.

Of course, this is not the *only* example where perception and physics are both important in understanding what we hear, there are many auditory "illusions."[7]

[6] H. Fletcher, "Loudness and Pitch of Musical Tones and Their Relation to this Intensity and Frequency," *Journal of the Acoustical Society of America* **6**, 56 (1934) (doi: http://dx.doi.org/10.1121/1.1915700); S. S. Stevens, "The Relation of Pitch to Intensity," *Journal of the Acoustical Society of America* **6**, 150 (1935) (doi: 10.1121/1.1915715).

[7] A good set of examples is available online from Diana Deutsch: `http://deutsch.ucsd.edu/psychology/pages.php?i=201` (the "octave illusion" is particularly good).

Chapter 1

Scales and Intervals

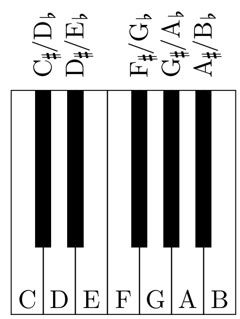
Before we even start to discuss how we should tune the frequencies of each note, we need to review, what *are* the notes? (At least, the standard notes in *Western* music.) This is standard stuff for musicians, but if you're not a musician, we'll get you up to speed here with the notation for the notes and the basics of how they're organized into scales.

Even more importantly, we need to know, how are the notes *related* to each other? Which notes sound *nice* together, and which ones don't? We'll take a first cut at these questions here, but leave a more in-depth discussion of **consonance** and **dissonance** to Chapter 7.

1.1 Western Chromatic and Diatonic Scales

Take a look at any piano-like instrument. The pipe organ in the photo below is a good example. What do you see? How are the keys (notes) arranged?

Nearly always, in a piano-like keyboard, you will see the following pattern of keys:



The keys are named according to the labels in the diagram. White (long) keys are labeled with the letters A–G (though it is conventional to start on C, as shown). The black (short) keys, on the other hand are named as *modifications* to the white keys, either **sharp** (♯) to indicate an *increased* pitch, or **flat** (♭) to indicate a *decreased* pitch. As you can see, this leads to some redundancy in names that we will discuss: C♯ and D♭ refer to the same note, for example, and such pairs of notes are called **enharmonic equivalents**. More generally, the sharp and flat modifiers just tell you to go up or down one key, respectively. Thus, for example, a B♯ is the same as a C, and an F♭ is the same as an E (multiple flats and sharps are also possible). To avoid ambiguity, it is possible to use the **natural sign** (♮) to indicate that the sharp/flat modifiers do not apply. That is, C is the same as C♮, and E♯ is the same as F♮.

This pattern of keys repeats over and over again on the piano keyboard. Each repetition belongs to a separate **octave**, and the keys in each repetition (octave) are named with the same letters. More precisely, consider two C keys, in adjacent repetitions of the pattern (adjacent octaves). These keys are said to be *one octave apart.* The octave is the most fundamental of the **musical intervals** that we will now discuss.

1.1.1 Musical Intervals

The "distance" between two musical pitches is a **musical interval**, and the all of the musical intervals have names. To understand the musical intervals, you first have to understand that *all of the keys on the piano, both black and white, are equivalent, as musical intervals go.* That is, going from one key to its immediate neighbor is always the same interval or change in pitch (which we will define more carefully later). So the interval between a C and C♯/D♭ is the same as the interval between E and F, and the interval between B and C.

This interval between adjacent notes is called the **semitone** (and also goes by the names **half step**, **half tone**, and **minor 2nd**). You can think of the "half" or "semi" as being half of the interval between the first two white keys, C and D. The white keys are of special importance, as we'll discuss shortly, and notice for now that they are *not* evenly spaced, because there not black keys between every pair of white keys.

The interval between the first two white keys, C and D, is the **whole tone** or **major 2nd**. Similarly, there are other whole-tone intervals between *any* next-door pair of white keys surrounding a black key, like F–G; or if there isn't a black key in between, you have to go up to the next black key to get a whole tone, like E–F♯. In general, you can deduce the intervals between C and other notes above it by counting the *number of white keys in the interval, including both the "beginning" and "destination" notes.* So the (major) 2nd goes from C to D (the beginning and end of the interval); the (major) 3rd goes from C to E and so we count C, D, E for three total notes. The table below shows the complete set of intervals in the piano octave shown.

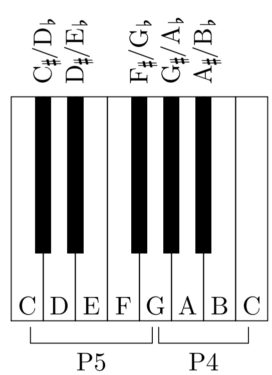
Note in C Major	**Interval**	**Interval Name**
C	P1	Perfect Unison
C♯/D♭	m2	Minor 2nd (Semitone)
D	M2	Major 2nd (Whole Tone)
D♯/E♭	m3	Minor 3rd
E	M3	Major 3rd
F	P4	Perfect 4th
F♯/G♭	A4/d5/TT	Augmented 4th/Diminished 5th (Tritone)
G	P5	Perfect 5th
G♯/A♭	m6	Minor 6th
A	M6	Major 6th
A♯/B♭	m7	Minor 7th
B	M7	Major 7th
C	P8	Perfect Octave

Note that the intervals involving the white keys are labeled by either **major** or *perfect*, while black keys are labeled with respect to a nearby white key—**augmented** with respect to a "perfect" note for a sharp note, and **minor** or **diminished** with respect to a "major" or "perfect" tone, respectively, for a flat note. Note that this identification only works when we are starting with C, but the pattern of piano keys is a useful visual reference for the pattern. Also, note that it is even possible to augment or diminish intervals that aren't "perfect" (e.g., an augmented major third here with respect to C is E♯, which on the piano is the F white key). However, the "basic" naming scheme for intervals is as in the table above.

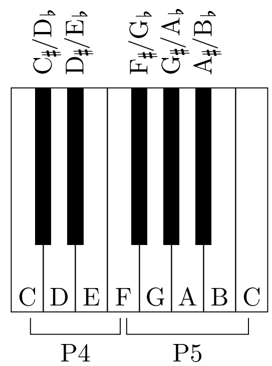
Generally speaking, the intervals are restricted to a single octave, because the names in other octaves just repeat. So an interval spanning more than an octave, for example, would be labeled by the in-octave interval plus an indication of the octaves. That is, a G is a perfect fifth above a C, and the *next* G is a fifth plus an octave above the same C. However, it is fairly common to refer to 9th (octave plus major second), 11th (octave plus perfect fourth), 13th (octave plus major sixth) intervals, especially in jazz chords.

1.1.1.1 Inversions of Intervals

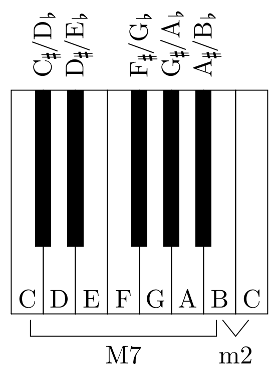
Intervals for notes *below* the C are a bit more complicated, because we have to understand the structure of the scale a bit better. For example, the G *below* C is a *fourth* down from C, not a fifth. This is best organized in terms of the **inversion** of an interval, which is the *complementary* interval that completes an octave. For example, a fifth is seven semitones, which accounts for all but five remaining semitones out of the twelve in an octave.



This is the same as a perfect fourth, and so the fourth is the inversion of the fifth. Conversely, the fifth is the inversion of the fourth, as you can see by doing the same counting on the piano keyboard.



Another good example is the major 7, which is all of the octave except for a semitone (minor 2), so the inversion of the major 7 is a minor 2:



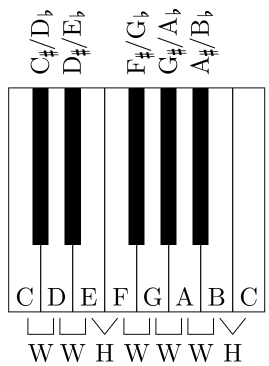
Again, conversely, the inversion of a minor 2 is the major 7. The table below summarizes the inversions of all the intervals.

Interval	**Inversion**
m2	M7
M2	m7
m3	M6
M3	m6
P4	P5
A4/d5/TT	A4/d5/TT

1.1.2 Chromatic and Diatonic Scales

Let's go back to the piano keyboard, and look at the arrangement of keys in one octave. There are twelve keys in each octave (seven white, five black), and the complete set of notes form the **chromatic scale**, representing the "full spectrum" of pitches in one octave. The notes in the chromatic scale are always separated by half tones (semitones).

Other scales *skip* notes in the chromatic scale. Recall again that there is a half-tone (semitone) interval between any two adjacent keys, and that two half tones make a whole tone. Thus, the *white* keys on the piano are spaced into either whole or half steps, depending on whether or not there is a black key in between. The diagram below is again of the piano keyboard in one octave starting on C (including the *next* C), with the whole-tone (W) and half-tone intervals between white keys marked.



The notes of the white keys,

C-D-E-F-G-A-B-C,

define the **C major scale**. The first and main note in this scale, C, is called the **tonic**, which is important in defining the starting point of the scale. Note that there are seven notes in this scale, not including the repeated octave. It is more fundamentally defined by the pattern of intervals

W-W-H-W-W-W-H.

So, for example, the F major scale is given by the notes

F-G-A-B♭-C-D-E-F,

which follows the same pattern of intervals, but starting and ending on F.

We can construct other scales by using the notes of the white keys, but starting on a note *other* than C. For example, starting on A gives the **A natural minor scale**,

A-B-C-D-E-F-G-A,

which is defined by the interval pattern

W-H-W-W-H-W-W.

Note that this is a **cyclic permutation** of the interval pattern of the major scale (cyclic permutations of ABCD are BCDA, CDAB, and DABC). There are five other possible cyclic permutations of this interval pattern, which give rise to yet more scales (called **modes**—the major scale is called the **Ionian mode**, and the minor scale is called the **Aeolian mode**)). Any seven-note-per-octave scale constructed this way is generally called a **diatonic scale**, which are important in Western music theory.

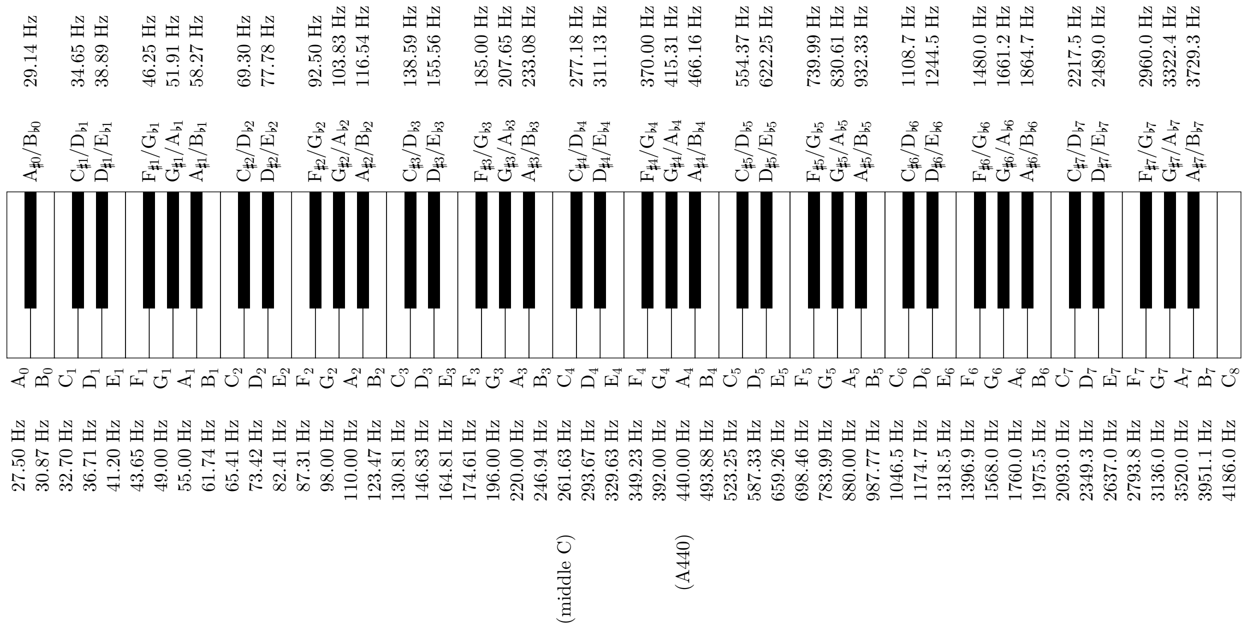
1.1.3 Musical Keys

A closely related concept to musical scales is that of a musical **key**. This is *not* the same thing as a *piano* key. The key of a song, or a section of a song, is something of a vague concept, so let's discuss this in terms of an example. If a passage is "in the key of C major," this can mean in the simplest sense that the piece uses only notes from the C major scale. In this case the distinction between key and scale is that the *key* refers to a group of notes, whereas the *scale* refers to an ascending or descending ordering of the same notes. The vague part comes when a piece of music uses **accidentals**, or notes outside the C-major scale (like, say, C♯). Using notes outside the defining scale doesn't mean that we have switched to a different key, but that either the notes outside the key are occasional exceptions, or the notes outside the key are to be understood within the context of the key.

In sheet music, one hint to the key of a piece is the **key signature**, which defines which notes are sharp or flat for the piece. However, the key signature does not uniquely specify the key; for example, C major and A minor have the same key signature, since they use the same notes (one being a cyclic permutation of the other). The possible key signatures are defined by how diatonic scales emerge from the circle of fifths. We will explain this in Section 2.1.

1.1.4 Musical Frequencies

Now recall that the octave on the keyboard that we've been working with is *repeated* several times—just over seven times, in fact. The full diagram of the piano keyboard appears below.



This diagram also includes the frequencies conventionally assigned to each musical pitch in contemporary music. This diagram assumes that the A pitch above middle C is tuned to 440 Hz (so that this reference pitch is referred to as "A440"). Actually, it turns out that *real* pianos are tuned very slightly differently from these frequencies (the high notes are a bit sharp, the low notes are a bit flat). This is called **stretched tuning**, where the piano octaves are tuned slightly wider than the $2:1$ ratio. The frequencies in the diagram are the "ideal" values, at least in **equal temperament**, as we will discuss in Chapter 3.

Where do these frequencies come from? Why are the piano keys organized into two types in the peculiar arrangement of the C major scale? Why are there twelve musical steps (semitones) per octave? These are all questions that we will answer soon.

1.2 Consonance and Harmonics

The first step in understanding the construction of musical scales and intervals is the notion of *what pitches sound nice when played together*? Musically speaking, a collection of pitches that sounds "nice" to most people is called a **consonance**, while a collection of pitches that sounds "tense" or "unpleasant" to most people is called a **dissonance**. Note that in music, dissonances are not necessarily *bad*—but often they are used to set up tension, which is then resolved by a subsequent consonance.

So where to we find consonances? The main hint comes from the harmonic series. Recall that for any periodic wave of frequency f_1, the **fundamental** frequency component f_1 is present, along with the harmonics, which are all the multiples $f_2 = 2f_1,\ f_3 = 3f_1,\ \ldots$ of the fundamental. Also recall that these harmonics occur in the sound of a vibrating string under tension, or the tone of a flute (i.e., a resonating, open tube). Most people regard these sounds as musically pleasing.

Thus, we can expect that *two pitches belonging to the same harmonic series will form a consonance.* That is, we might expect a 100 Hz tone to sound nicely when combined with a 200 Hz or a 300 Hz tone, being the respective second and third harmonics of the fundamental. We might also expect two tones to sound nicely together even if one of them isn't the fundamental: for example, 200 Hz and 300 Hz tones should sound well together, since they are both harmonics of 100 Hz. To state this another way, we might expect any two pitches f and f' to form a consonance if they are multiples of the same fundamental; that is, if we can write $f = mf_1$ and $f' = nf_1$, where f_1 is some fundamental frequency, and m and n are whole numbers (integers). Then the *ratio* of the two frequencies is

$$\frac{f}{f'} = \frac{mf_1}{nf_1} = \frac{m}{n}, \tag{1.1}$$

after canceling the fundamental frequency. A ratio or fraction that can be written in terms of only whole numbers is a **rational number**. (Note that we can also say that the ratio of f to f' is $m : n$, or "m to n.") So our rule is as follows: **two pitches form a consonance if the ratio of their frequencies is a rational number**.

A couple of caveats are in order here. First, two pitches will only *really* form a consonance if the frequency ratio is a *simple* rational number—that is, a ratio $m : n$, where m and n are *small* integers. Thus, two pitches with a frequency ratio of 2 : 1 or 3 : 2 should produce a harmonious sound. However, if we can only write the ratio in terms of *large* integers, like 218 : 191, we might not expect the pitches to form a consonance. This is consistent with our harmonic argument; while we expect to hear the harmonic series of a plucked string, we will only hear the *first few* harmonics. For a typical musical pitch of 440 Hz (A440), the 218th harmonic registers at about 96 kHz, well beyond the range of human hearing. So very high harmonics (and consequently, complex ratios) don't fit into this argument for consonance. Similarly, if the ratio can't be written as a ratio of integers at all—that is, the number is *irrational* (irrational numbers include π, e, and $\sqrt{2}$)—then we might also not expect a consonance.

The other thing to note is that if the ratio of two pitches is *very close to a simple rational number*, then we also expect a consonance. That is, 300 Hz and 200 Hz tones form a ratio of 3 : 2, and therefore form a consonance. But what about 300 Hz and 200.1 Hz? Technically the ratio of these two frequencies is 3000 : 2001, which is not very simple. But it will still sound consonant, because the ratio is *close* to 3 : 2. Our ears can tolerate a bit of "slop" in the tuning of frequencies, which compensates for the mismatch in the ratio. Another example of this is when several of the same instruments (say, a trumpet section) sound the same note together. They will not play *exactly* the same pitch; there is always a tiny bit of variation due to embouchure, breathing, and so on. But as long as the pitches match pretty well, the ensemble will still sound good, and in fact the slight pitch mismatch cues your ear, telling you that you are listening to more than just a single trumpet player. On the other hand, if the frequencies mismatch *too* much, then the effect isn't as nice—like, well, an out-of-tune brass section.

In the sixth century BC, Pythagoras had already discovered that pitches with simple ratios form consonances, through experiments with strings. In these experiments, he sounded together two strings of different lengths, but otherwise identical in composition, diameter, and tension. Actually, he did this by dividing a *single* string under tension, stretching the string over a movable bridge that isolated the vibration of two parts of the string from each other. He found that the pitches of the two strings were consonant whenever the *ratio of the lengths* formed the ratios 1 : 1, 2 : 1, 3 : 2, and 4 : 3. But we know from Mersenne's laws that the frequency of a string is proportional to the inverse of the length of a string, so the corresponding frequency ratios are 1 : 1, 1 : 2, 2 : 3, and 3 : 4. (Note that the ordering of the numbers in the ratio isn't important in determining its "level of consonance.") This agrees with our expectations based on harmonics, but the harmonic argument is more general, for example predicting that a ratio of 5 : 4 should also be consonant.

1.2.1 Examples

Let's take a break for a few listening examples. The following sound clips are pairs of pitches played as asymmetric triangle waves (so all harmonics are present). The lower pitch is middle C (261.63 Hz), and there is a higher pitch also played to demonstrate the consonance of the interval.

As a basis for reference, the middle C sounds alone (i.e., a unison interval).

The simplest ratio for an interval is 2 : 1, for the perfect octave, which sounds very consonant:

The perfect fifth (3 : 2) is also very consonant,

as are the perfect fourth (4 : 3),

and the perfect sixth (5 : 3):

Those are all consonant intervals, being simple ratios. What about a *dissonant* interval? As we mentioned *irrational* ratios should dissonant, provided they are not very close to a simple, rational number. One ratio that turns out not to be especially close to any simple ratio is the **golden ratio**, $(\sqrt{5}+1)/2 \approx 1.61803$, and this interval sounds quite dissonant:

1.2.2 Octaves

A special musical interval that we have mentioned is the **octave**, or **perfect octave**. We can define the octave as the interval corresponding to a frequency ratio of 2 : 1. This is, in fact, the *only* musical interval that we can define *unambiguously* this way; the other ones, it turns out, will depend on a choice of convention—something we will discuss at length.

If a frequency f' is double the frequency f (i.e., $f' = 2f$, or the ratio $f' : f$ is 2 : 1), then f' is said to be *one octave above* f. If a frequency f' is *half* of the frequency f (i.e., $f' = f/2$, or the ratio $f' : f$ is 1 : 2), then f' is said to be *one octave below* f.

This ratio is about as simple as it gets (the only simpler one is perfect unison, or 1 : 1, but that's just the same note played twice), and as we might expect, this interval is consonant. So consonant, in fact, that the pitches almost sound like the same pitch, but with a different tone quality. This is reflected in the naming of the notes: musical pitches that are separated by perfect octaves *go by the same name*. So any pitch that is an octave above or below a C (or multiple octaves above or below), is *also* called a C. In the piano diagram above (on p. 24), the different octaves are distinguished by different subscripts. So for example, C_3 is one octave above C_2, and thus, the frequency of C_3 is double that of C_2. This notation is called **scientific pitch notation**.

As a demonstration, listen to the following sound clip, which is the same note played through 9 octaves (A_0 through A_8), played as a triangle wave).

Now listen to a similar ascending note sequence that starts on A_0 and ends on A_8, but *misses* the octaves in between by one or two semitones in either direction.

Which sequence of notes sounds much more like the same note, but played with different tone qualities?

1.3 Exercises

Problem 1.1

Which of these frequencies should sound "nice" (consonant, or not dissonant) when played together with 250 Hz?

(a) 200 Hz

(b) 60 Hz

(c) 270 Hz

(d) 183.28 Hz

(e) $(100 \times \pi)$ Hz

Problem 1.2

Suppose you construct a diatonic scale in the same way as the major or minor, but start on E and stay on the white keys. What is the pattern of whole and half steps that defines this mode?

Chapter 2

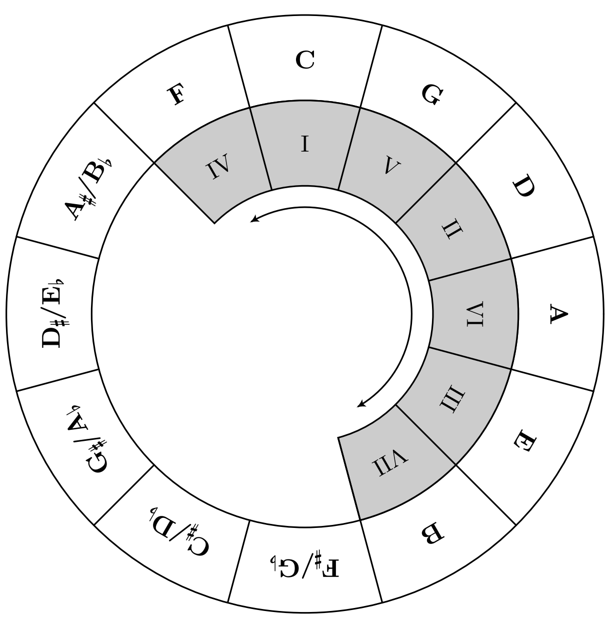
Pythagorean and Just Scales

We ended the review of musical intervals in Chapter 1 by noting that pitches separated by octaves—frequencies in a 2 : 1 ratio—sound so similar that in some sense they aren't even really different notes. In fact, the prevailing convention is to use the same letter to label pitches separated by octaves. So while it would seem to be pretty sensible to preserve the notion of an octave in any musical scale, octaves *by themselves* are not a sufficient basis for anything but very bland music.

However, the *next* most consonant interval—a ratio of 3 : 2, or the **perfect fifth**—can be used as the basis to construct musical scales. This construction is attributed to Pythagoras (although Pythagoras' writings no longer survive to act as primary sources), and is still reflected in the structure of the Western diatonic and chromatic scales in common use today.

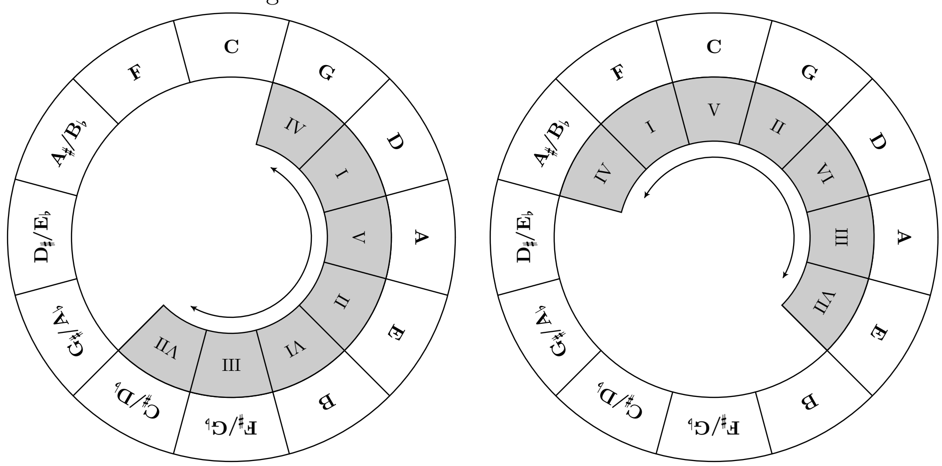
2.1 Circle of Fifths

An important tool to visualize and organize the construction of scales is the **circle of fifths**. The basic idea is to organize the twelve notes of the chromatic scale in a circle, such that each note has its relative fifth as its neighbor in the clockwise direction.



By convention, we will put the C note at the twelve-o'clock position. Then G is the fifth in the C-major scale, so it appears at the one-o'clock position. Continuing, D is the fifth in the G-major scale, so it appears at two o'clock, and so on.

Going back to the C-major scale of Section 1.1.2, we can see if we label the notes from I-VII according to their position in the scale, we can see that they cluster in the range of eleven o'clock to five o'clock. And remember that by the way we constructed the circle as a sequence of fifths, this diatonic structure should hold elsewhere in the circle. Hence, think of the shaded major-scale markers as a movable "slider" to find the notes of different scales. Let's consider two examples: one where we rotate the slider two "notches" clockwise, and one where the slider goes one notch counter-clockwise:



In the first (left-hand) case, the **tonic** (I) of the scale is on D, so the notes now mark the D-major scale. Writing these out in order, the D major scale is then

D-E-F$\sharp$-G-A-B-C$\sharp$-D.

In the second (right-hand) example, the tonic is F, so the F-major scale is

F-G-A-B$\flat$-C-D-E-F.

Note that for tonics clockwise of C, it is conventional to represent the altered pitches with sharps (that is, F$\sharp$ appears in D major, not G$\flat$), while counter-clockwise tonics use flatted notes (hence B$\flat$ appears in the F-major scale, not A$\sharp$). From this same method, we can also read off the major scales or keys with one, two, three, and four sharps (G, D, A, and E, respectively), and major scales or keys with one, two, three, and four flats (F, B$\flat$, E$\flat$, and A$\flat$, respectively).

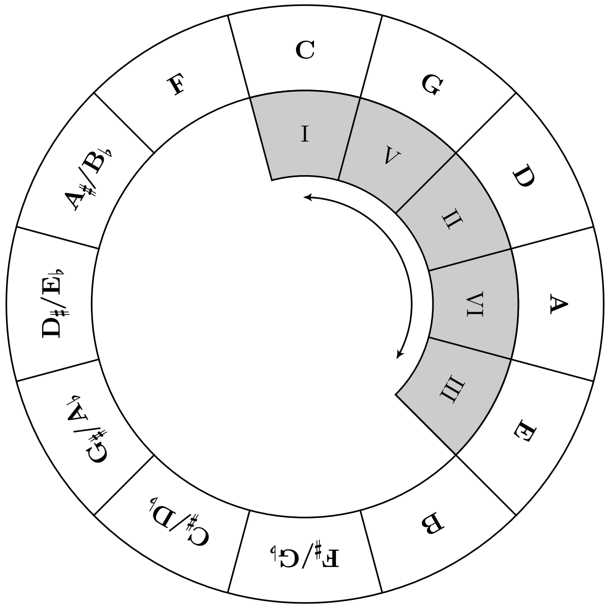
2.2 Pentatonic Scale

Now to proceed with the construction of Pythagorean scales. We have to pick a tonic pitch as the basis of the scale, and we will choose C here (again, this is an arbitrary choice, which we can change just by rotating the circle). We have to choose some frequency for this (say, 261.63 Hz for middle C, but even this is arbitrary) In the circle of fifths, the next clockwise note (G) is a perfect fifth above C, so we define G so that the frequency ratio of G to C is $3:2$.

The next step is just to continue this procedure for the next note. That is, the ratio of D to G is $3:2$, being the next fifth on the circle. But we want to reference everything to the tonic C. The frequency ratio of D to C is just the product of the ratios for D to G and then G to C, or $(3/2)^2 = 9/4$, because we multiplied by 3/2 twice to go two steps (fifths) along the circle. But we're not quite done yet: $9/4 = 2.25$ is bigger than 2, which means this pitch is higher than the C one octave *above* our original tonic. But we want to construct the notes of a scale *within* a single octave. So we have to go down by octaves until we get the note in the right range (i.e., a ratio between 1 and 2). Dividing by 2 to go down one octave gives us a ratio of $9/8 = 1.125$, which is in the right range.

And we can continue this pattern. The next fifth is A, which is a ratio $(3/2)^3 = 27/8$ above C. But one octave down is $27/16 = 1.6875$, which is in the right range. And the next fifth beyond that is E, given by a frequency ratio of $(3/2)^4 = 81/16$, and we have to go down *two* octaves to $81/64 = 1.2656$ to end up within the first octave above the tonic.

So far, we have set up the pitches of the first five notes in the circle of fifths, moving clockwise from the tonic, as shown:



These five notes determine the **major pentatonic scale** (*penta*=five, *tonic*=note), a scale that is fairly simple but in common use in popular Western music. We can summarize our calculated frequency ratios, including the octave, and the musical interval names we defined before, thusly:

Note in C Major	Interval	Pythagorean Frequency Ratio	
C	P1	1/1	1.0000
D	M2	9/8	1.1250
E	M3	81/64	1.2656
G	P5	3/2	1.5000
A	M6	27/16	1.6875
C	P8	2/1	2.0000

Here we are only keeping track of the frequency ratios, or the frequencies of the notes *relative to the tonic* (C, in this case). That is, once we fix the *absolute* frequency of C (say, 261.63 Hz), the other frequencies follow by multiplying the tonic frequency by the corresponding ratio [e.g., D is $(9/8)\times 261.63$ Hz $= 294.33$ Hz]. But the frequency ratios are unchanged (by our construction here), even if the tonic is tuned to a different absolute frequency. The black keys on the piano form a pentatonic scale, for example, in particular the F♯ major pentatonic scale.

2.2.1 Significance of the Pentatonic Scale

A number of simple but well-known melodies can be played in the pentatonic scale (a famous example is the traditional Scottish folk song that in the 18th century became *Auld Lang Syne*[1]). The pentatonic scale even appears to be fundamental to the human experience. For example, see Wulf Hein demonstrating a 40,000-year-old paleolithic bone flute, tuned to a pentatonic scale, in the Herzog documentary *Cave of Forgotten Dreams*;[2] or witness Bobby McFerrin in a great interactive demonstration of the power and universality of the pentatonic scale that all audiences "get."[3]

[1] Modern/Phish example: `http://www.youtube.com/watch?v=SuH16Q8c3sY`
[2] `http://www.youtube.com/watch?v=yUCBBDV2Tzk`
[3] `http://www.youtube.com/watch?v=ne6tB2KiZuk`

The pentatonic scale also universal in the sense of showing up in many cultures. It is common in East-Asian music—for example, the Japanese **shakuhachi** flute is traditionally tuned to play a pentatonic scale.[4] Similarly, the pentatonic scale was known in ancient China, being described in texts since about the fourth century BC. The Chinese constructed the pentatonic scale (and the chromatic scale that we will get to in Section 2.4) in basically the same way as the Pythagorean construction we outlined. The "*wusheng*" ("five tones") were named *gong*, *shang*, *jue*, *zhi*, and *yu*; and, for example, great significance in terms of the status of the nation was attached to these tones.[5]

It shows up in indigenous Latin-American music, particularly Andean music, although its origin—whether from Incan tradition or European influence—is uncertain.[6] A good example of Latin-American practice is the **Julajula** pan pipes of the Aymara people of the *Altiplano* (high plateau, mainly in Bolivia and Peru). These pipes come in sets of two, and the notes of the pentatonic scale alternate between the two sets.[7] West-African music during the 18th and 19th centuries—an important component of the musical ancestry of American jazz—also made heavy use of pentatonic scales.[8] We already mentioned the pentatonic scale in Scottish folk music, but it was also more broadly important in European folk music ("Amazing Grace" is a famous example).

2.3 Diatonic Scale

The Pythagorean construction of diatonic scales—in particular, the C-major scale—then proceeds in a similar way, and Pythagoras is indeed credited as being the first to construct the full diatonic scale. To add in the extra two notes, we extend the pentatonic scale that we just constructed, but *by one note in each direction.* To see what we mean, first we add in the B, which is a fifth above the E. This is a total frequency ratio of $(3/2)^5$ with respect to the tonic, which is the same as 243/128, once we lower it to the correct octave.

We add the last note, F, by proceeding *counter-clockwise* from the tonic. That is, C is a perfect fifth above F, so the frequency ratio of C to F is 3/2. But we want to reference everything to the tonic, so the frequency ratio of F to C is 2/3. This is less than 1, so we need to raise it by an octave to a ratio of 4/3. This is the ratio corresponding to the perfect fourth: recall that the fourth is the inversion of the fifth, and we can see that explicitly here, because multiplying the ratios for these intervals give the ratio for the octave: $(3/2) \times (4/3) = 2$. (Remember that **adding** or **combining** two musical intervals, here the fifth and fourth, is the same as **multiplying their ratios**.)

The full diatonic scale (specifically, C major) is summarized in the table below, with the two added notes highlighted. Remember that to stay in one octave, we had to multiply or divide by factors of 2 to end up with a ratio between 1 and 2.

Note in C Major	**Interval**	**Pythagorean Frequency Ratio**	
C	P1	1/1	1.0000
D	M2	9/8	1.1250
E	M3	81/64	1.2656
F	P4	4/3	1.3333
G	P5	3/2	1.5000
A	M6	27/16	1.6875
B	M7	243/128	1.8984
C	P8	2/1	2.0000

[4]Nice example of a shakuhachi performance: `http://www.youtube.com/watch?v=f7s-wXZWT5o`.

[5]Scott Cook, "'Yue Ji' – Record of Music: Introduction, Translation, Notes, and Commentary," *Asian Music* **26**, No. 2 (1995), p. 1 (doi: 10.2307/834434).

[6]Barbara Bradby, "Symmetry around a centre: music of an Andean community," *Popular Music* **6**, 197 (1987) (doi: 10.1017/S0261143000005997).

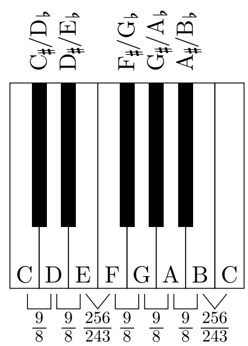
[7]Max Peter Baumann, "Music and Worldview of Indian Societies in the Bolivian Andes," in *Music in Latin America: An Encyclopedic History*, Malena Kuss, Ed. (U. Texas Press, 2004), vol 1, p. 101 (ISBN: 0292702981).

[8]James Lincoln Collier, "Classic jazz to 1945," in *The Cambridge History of Twentieth-Century Music*, Nicholas Cook and Anthony Pople, Eds. (Cambridge, 2004), p. 123 (doi: http://dx.doi.org/10.1017/CHOL9780521662567).

Note that we have taken the *names* of the notes to be *given*, but of course the construction had to originally go the opposite way: The notes were first constructed according to the chain of fifths, and then once reduced to the same octave and sorted out in order of pitch, *then* they can be named in alphabetical order.

2.3.1 Pythagorean Whole Tone and Diatonic Semitone

Now look at the intervals between adjacent notes in the diatonic major scale. We can actually see from the circle of fifths in Section 2.1 that adjacent notes are either two fifths apart (F-G, C-D, G-A, D-E, A-B), or six fifths apart (F-E, C-B). There are therefore only two intervals for adjacent notes in the Pythagorean scale. The whole tones are separated by a ratio of 9/8, the same interval as C-B. The semitones are related by the C-B interval of 243/128, which we must invert to 256/243 to obtain the B-C interval.



Thus, the ratio of 9/8 is the **Pythagorean whole tone**, and the ratio 256/243 is the **Pythagorean diatonic semitone**. Note that two Pythagorean diatonic semitones, $(256/243)^2$, is *not* equivalent to a Pythagorean whole tone. This already points to some of the problems with Pythagorean tuning that we are coming to.

Just to make sure you're following the piano diagram above, let's do a quick example.

Example

Problem. What is the ratio ("difference") between the Pythagorean D and E notes?

Solution. From the table above for the Pythagorean diatonic scale, the ratios for D and E are 9/8 and 81/64, respectively, compared to the tonic. To get the ratio from D to E, we can just divide the ratios, in particular divide the bigger E ratio by the smaller D ratio: $(81/64)/(9/8) = 9/8$. That is, the ratio of E to D is 9/8, in agreement with the piano-key diagram above.

2.4 Chromatic Scale

The Pythagorean scale construction that we have outlined so far was a fine way of constructing a diatonic scale: by chaining fifths, at least various pairs of notes were guaranteed to sound nicely together, even if others won't necessarily. However, for modern music, a single scale isn't enough. Different compositions are in different keys, and it is common to change keys within a single song to make it more harmonically interesting. Therefore we need to continue the construction to include the sharps and flats, and thus the complete chromatic scale.

To do this, we can go back to the circle of fifths in Section 2.1, and (by a somewhat arbitrary convention) we will extend the chain of fifths clockwise to F♯(G♭) at the 6 o'clock position, and generate the remaining notes by extending the chain of fifths *counter*-clockwise from F to D♭(C♯) at the seven o'clock position. Note that we emphasize the clockwise chaining by calling the note by its sharp name, and the counter-clockwise chaining by referring to the flat name, though we are also including the harmonically equivalent note name as well in parentheses. The process of chaining fifths is the same, and the results are summarized in the table below, with the new notes highlighted.

Note in C Major	Interval	Pythagorean Frequency Ratio	
C	P1	1/1	1.0000
$D_\flat(C_\sharp)$	m2	256/243	1.0535
D	M2	9/8	1.1250
$E_\flat(D_\sharp)$	m3	32/27	1.1852
E	M3	81/64	1.2656
F	P4	4/3	1.3333
$F_\sharp(G_\flat)$	A4/d5/TT	729/512	1.4238
G	P5	3/2	1.5000
$A_\flat(G_\sharp)$	m6	128/81	1.5802
A	M6	27/16	1.6875
$B_\flat(A_\sharp)$	m7	16/9	1.7778
B	M7	243/128	1.8984
C	P8	2/1	2.0000

As before, the continued notes result in values that involve fractions with increasingly large numbers, and we might expect these to be increasingly less consonant with the tonic. Note that adding in the extra notes, we have a semitone from F (4/3) to $F_\sharp$ (729/512) that is defined by the interval

$$\frac{729/512}{4/3} = \frac{2187}{2048} \approx 1.0679, \tag{2.1}$$

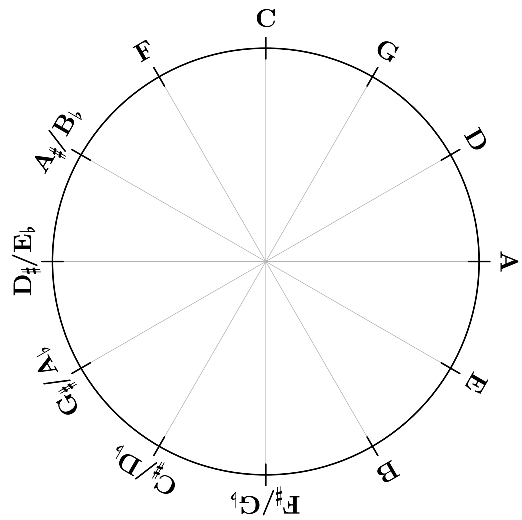
(Pythagorean chromatic semitone)

which is slightly larger than the usual Pythagorean diatonic semitone $256/243 \approx 1.0535$. This oddball semitone is called the **Pythagorean chromatic semitone**, and is related to the Pythagorean comma below.

2.5 Pythagorean Wolf Fifth

At this point we have managed to fill in all of the pitches in the chromatic scale by chaining together fifths, and it would seem that this wraps up the whole business of musical scales. Unfortunately, things don't quite work out this neatly, and so we are far from done.

The problem is that the cycle of fifths *repeats*, but the *chain* of fifths *doesn't*. To visualize this, recall again the circle of fifths. Because the cycle of fifths repeats, we can think of the notes of the chromatic scale as partitioning the octave into twelve equal "slices," as shown here.

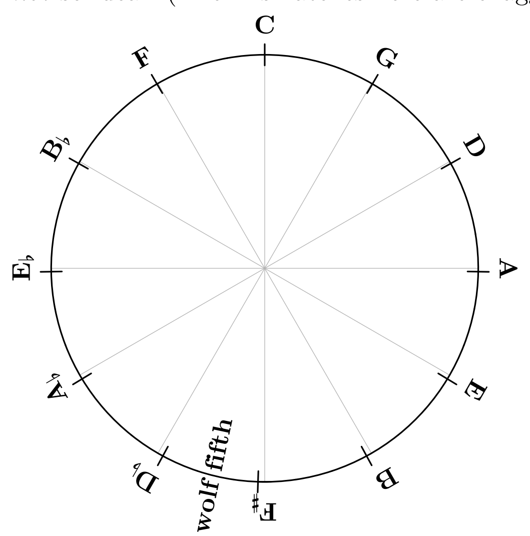


However, as it turns out, this *isn't* what happened. By extending the chain clockwise to F♯ and counter-clockwise to D♭, the interval between these two notes is the ratio of $2 \times (256/243)$ to $(729/512)$, where we moved the D♭ up an octave, for a total of

$$R_{\text{wolf}} = \frac{262\,144}{177\,147} = \frac{2^{18}}{3^{11}} \approx 1.4798. \tag{2.2}$$

(Pythagorean wolf-fifth interval)

This is quite a bit less than the ideal ratio of $3/2 = 1.5$, and so the requirement of chaining ideal fifths together leaves one fifth that is *not* so ideal. (The mismatches here are exaggerated for dramatic effect!)



This out-of-tune fifth is called a **wolf fifth**, because of its unpleasant, howling quality. Compare two asymmetric-triangle-wave tones (all harmonics present) sounding a *just* perfect fifth,

with a Pythagorean wolf fifth:

The wolf fifth is considered unpleasant enough that it should be avoided or at least hidden in compositions. This can be done when playing in a single key, but once an instrument is tuned for a particular key, avoiding the wolf fifth places severe constraints on the available alternate keys to be played on the instrument without retuning (and thus, the available keys within a single composition).

2.5.1 Pythagorean Comma

In other words, this happens because the ideal ratio for the perfect fifth is 3/2. But if we chain together a bunch of these, ideally after a chain of 12 we would be back to the same note. But 12 of these intervals gives a total interval

$$\left(\frac{3}{2}\right)^{12} \approx 129.75. \tag{2.3}$$

This is *almost*, but not *quite*, the same as seven octaves:

$$2^7 = 128. \tag{2.4}$$

The ratio of these two intervals, representing the mismatch of the chain of fifths with the chain of octaves, is

$$\frac{(3/2)^{12}}{2^7} = \frac{3^{12}}{2^{19}} = \frac{531\ 411}{524\ 288} \approx 1.0136, \tag{2.5}$$

(Pythagorean comma)

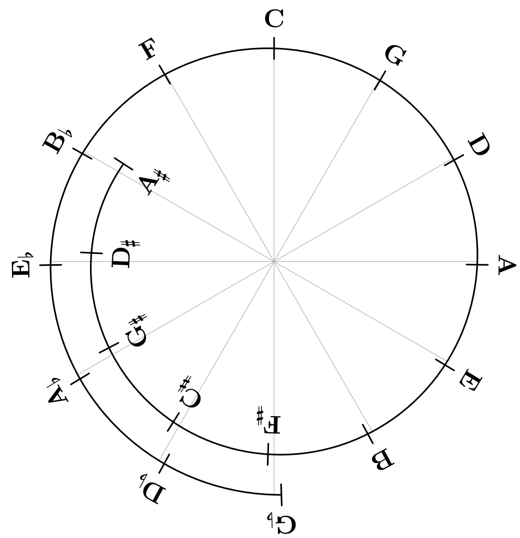
and is called the **Pythagorean comma** or **ditonic comma**. This is the same as the ratio of the ideal fifth (3/2) to the Pythagorean wolf fifth (R_{wolf}). To explain the terminology: A musical **comma** refers to a small difference, discrepancy, or interval in pitch. The **ditonic** here refers to **Ptolomy's diatonic diatoniaion**, or his **ditonic scale** (the same as Pythagoras' scale), where the "ditone" is the major third, consisting of two identical (9/8) whole tones.[9]

Note that the Pythagorean comma *isn't* a result of a bad choice of chaining, say, 12 fifths together to return to the same pitch (give or take a few octaves). It's not hard to argue that there will *always* be some comma no matter how many fifths we use. A chain of n fifths has a ratio of $3^n/2^n$, and reducing this by m octaves gives a ratio of $3^n/2^{n+m}$. But this can never be one, because it is an odd number divided by an even number (a ratio equal to one is the ratio of two *identical* numbers). It is in principle possible to obtain a *smaller* comma by chaining together more fifths—in fact, the comma may be made as small as you like, though at the expense of an unwieldy extra set of notes in the chromatic scale. For example, the next chain of fifths that works well is 41 (see Chapter 8), which is close to 24 octaves. This choice gives a *slightly* smaller comma than the Pythagorean comma (a 1.15% error vs. a 1.36% error), but with more than four times as many notes in the octave. So, as it turns out, a chain of 12 fifths is a rather *clever* choice: it is economical in terms of notes, while keeping the comma reasonably small.

2.5.2 Spiral of Fifths

What all this means is that for a given note, such as C♯/D♭, the pitch depends on which way we go around the circle of fifths, since the clockwise construction (C♯) is different from the counter-clockwise construction (D♭). In this sense, we can think of the circle of fifths as more of a *spiral* of fifths.

[9] J. Murray Barbour, *Tuning and Temperament: A Historical Survey*, 2nd ed. (Michigan State College, 1953) (ISBN: 0486434060), p. 21. First edition available online at `https://archive.org/details/tuningtemperamen00barb`.

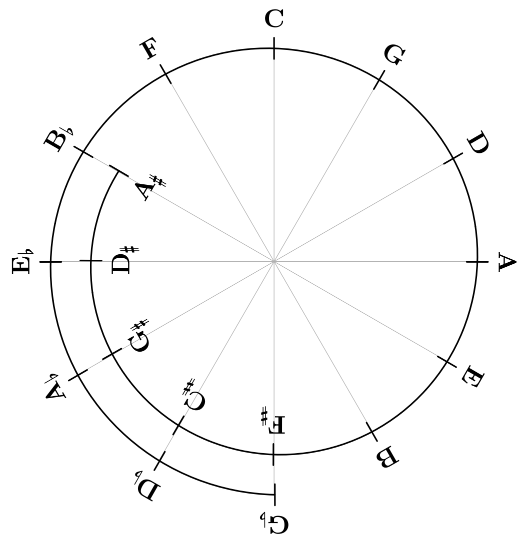


Therefore, in keys that use sharp or flat notes, you can think of enharmonically equivalent notes like C♯ and D♭ as actually being slightly different, if the ideal fifths need to be maintained in all keys. This is implemented in **enharmonic keyboards** for keyboard instruments (like pianos and organs), for example, by splitting the black keys of the piano keyboard into two keys, one for the sharp, and one for the flat.[10] In this way it is possible to play in multiple keys while avoiding wolf fifths. This happens by literally using different sets of (physical) keys when playing in different (musical) keys. However, such keyboards are not in common use due to the extra complications of having yet more keys, and because equal temperament (coming up in Chapter 3) offers a much simpler, if somewhat compromised, solution. Such considerations apply also to other instruments: some fingering charts for the recorder, for example, give different fingerings for sharp and flat notes that are otherwise enharmonically equivalent.[11] Also, on fretless string instruments (violin, cello, double bass), slightly different fingerings produce different enharmonic notes; on a double bass, the difference in enharmonic fingerings can exceed 2 cm (Problem 4.3). Note that the precise pitch of the enharmonically equivalent sharp and flat depends on the exact tuning scheme used, however. In Pythagorean tuning, we can see from the spiral that, say G♯ is slightly *higher* in frequency than A♭. However, in the more important quarter-comma meantone temperament (coming up in Chapter 4), this is reversed: A♭ is a bit *sharper* than G♯. In equal temperament (described below), G♯ and A♭ are exactly the same.

The above spiral diagram actually has a greatly exaggerated misalignment to make it clearly visible. When drawn to proper scale in Pythagorean tuning, the misalignment is barely visible, as shown below.

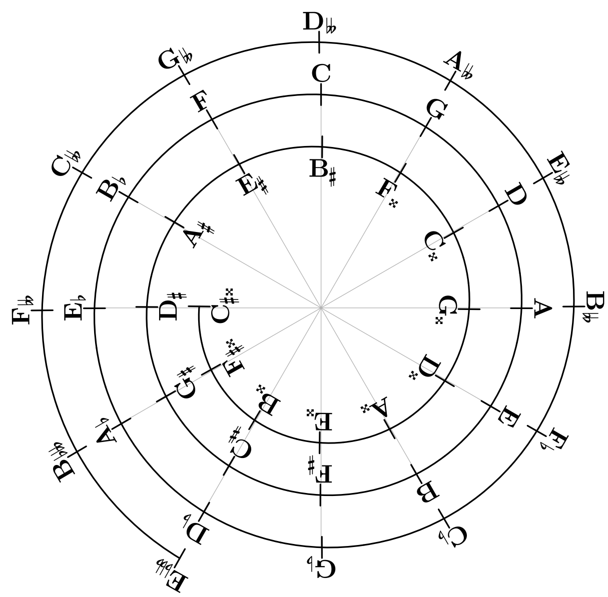
[10]For a good example, of an enharmonic keyboard, see `http://www.flickr.com/photos/40922198@N00/396791619`

[11]see, e.g., the fingering for the C tenor recorder in a pamphlet by Thomas Stanesby, Jr., "A New System of the Flute Á Bec or Commmon English Flute" (c. 1732), reprinted at `http://www.flute-a-bec.com/textestanesbygb.html`, and fingering chart reproduced at `http://www.flute-a-bec.com/tablstanesbygb.html`.



However, the ear can sense beats of closely spaced pitches, so in an auditory sense this misalignment is much more important.

Note that we can continue this spiral construction as far as we like, as illustrated below (and drawn to scale in Pythagorean tuning), by extending musical pitches to multiple sharps and flats.



The notation for extra sharps and flats here includes the double-sharp (𝄪), triple-sharp (♯𝄪), double-flat (𝄫), and triple-flat (♭𝄫) symbols. The idea is to repeat the sequence of fifths F-C-G-D-A-E-B, but to add a sharp for each clockwise repetition (i.e., when moving up by fifths), and to add a flat for each counter-clockwise repetition (i.e., when moving down by fourths). Then, for example, B♭𝄫, A♭, G♯, and F♯𝄪 are all enharmonically equivalent notes, but have slightly different frequencies—the differences are represented by the Pythagorean comma—in Pythagorean tuning.

2.6 Pythagorean Major Third

Another serious issue with Pythagorean tuning is that the major third is not ideal. The Pythagorean scale is built up from ratios involving threes and twos. So the consonant ratio 2 : 1 appears as the octave, and the next most consonant ratios 3 : 2 (perfect fifth) and 4 : 3 (perfect fourth) appear naturally. But the ratio

5 : 4 never appears, and *can't* appear, because there is no way to combine threes and twos to obtain a five. The closest ratio to $5/4 = 1.25$ is the major-third interval, C-E, at $81/64 \approx 1.2656$. But this is much less consonant than 5/4, because the ratio involves much larger numbers.

Listen for yourself—the ratio of 5/4 sounds very smooth,

whereas the sharper Pythagorean ratio of 81/64 has a rough sound that many musicians find objectionable:

The interval or "error" between the ideal or **just** major third (5/4) and the Pythagorean major third (81/64) is

$$\frac{81/64}{5/4} = \frac{81}{80} = 1.0125, \tag{2.6}$$

(syntonic comma)

and is called the **syntonic comma** or the **comma of Didymus**. To explain the odd terminology here: *Syntonic* means "tightly stretched," and refers to **Ptolomy's Diatonic Syntonon**, in comparison to his **Diatonic Malakon** (malakonic = soft). Ptolomy's syntonic scale contains a 5/4 major third, as does Didymus' diatonic scale; hence the name of the comma.[12]

The difference between the thirds is clear to trained ears. Hermann von Helmholtz noted, for example,[13]

> It is not at all difficult to distinguish the difference of a comma $\frac{81}{80}$ in the intonation of the different degrees of the scale, when well-known melodies are performed in different 'colourings,' and every musician with whom I have made the experiment has immediately heard the difference. Melodic passages with Pythagorean Thirds have a strained and restless effect, while the natural Thirds make them quiet and soft, although our ears are habituated to the Thirds of the equal temperament, which are nearer to the Pythagorean than to the natural intervals.

(More on equal-temperament thirds to come.) So while the Pythagorean scale is an elegant construction, it is only suited to music based on fifths (and only particular fifths!), as in the Middle Ages. For more complex music involving other intervals, it is less ideal, and so other systems of tuning superseded it.

2.7 Just Intonation

The "ideal" musical intervals we have referred to, where the frequencies are related by ratios of small whole numbers—that is, based on the natural relations defined by harmonics—are called **just intervals** (also **pure intervals**), and a musical tuning of a scale based on just intervals is called **just intonation**. We can go through the just intervals of the chromatic scale to establish our ideal "target" intervals. As we have already seen, the Pythagorean tuning hits some of the intervals and misses others. As we will see, this wasn't due to a lack of cleverness on the part of Pythagoras or the ancient Greeks; rather it is inherent in *any* system of tuning.

2.7.1 Just Ratios

The fundamental intervals of the unison (1 : 1) and the octave (2 : 1) are so important that they are fixed in every system of tuning. We have mentioned the just perfect fifth, a ratio of 3 : 2. Remember that we have defined the concept of the inversion of the interval: an interval combined with its inversion form an octave. So if α is the ratio for an interval, and β is the ratio of the inversion, then $\alpha\beta = 2$, an octave. Then the inversion ratio is $\beta = 2/\alpha$. For example, the inversion of the perfect fifth is the perfect fourth, and by

[12] J. Murray Barbour, *op. cit.*

[13] Hermann L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, second English edition translated by Alexander J. Ellis (Dover, 1954), corresponding to the fourth German edition (1877, with notes and additions to 1885), p. 266 (ISBN: 0486607534).

the definition we have just deduced, the ratio for the perfect fourth is $2/(3/2) = 4/3$, as we have already deduced. The next simplest just interval is $5 : 4$, which we have noted is the major third. The inversion of the major third is the minor sixth, and the ratio is therefore $2/(5/4) = 8/5$. Continuing in this way, the next simplest ratio is $6 : 5$ (or 1.2), which is close to the Pythagorean minor third, and we will take this to be the just minor third. The inversion is the major sixth, with a ratio of $5 : 3$, another nice simple ratio.

Continuing in the ratio of fifths, we have a ratio of $7/5 = 1.4$, which defines the just tritone. Note that the inversion of the just tritone is $10/7 \approx 1.429$, which is not quite the same; we will not take this inversion to be part of the just scale.

The remaining intervals are dissonant, and so their just ratios are not quite so critical. For the major second or whole tone, we can take the ratio 9/8. This ratio corresponds to the ninth harmonic (reduced by three octaves), and it is the sam ratio we obtained in the the Pythagorean scale. This is also the difference (musical difference, i.e., *ratio*) between the perfect fifth (3/2) and the perfect fourth (4/3):

$$\frac{3/2}{4/3} = \frac{9}{8}. \tag{2.7}$$

Other similar ratios are possible for the major second. For example, 10/9 arises as the difference between the perfect fourth (4/3) and minor third (6/5):

$$\frac{4/3}{6/5} = \frac{10}{9}. \tag{2.8}$$

The inversion of the major second (9/8) is the minor seventh (16/9). Again, other intervals are possible from the minor seventh such as 7/4 (the **septimal minor seventh** or **harmonic seventh**).

As the semitone or minor second, we can take the difference between the perfect fourth (4/3) and major third (5/4):

$$\frac{4/3}{5/4} = \frac{16}{15}. \tag{2.9}$$

The inversion is the major seventh (15/8), so this is also the interval between the major seventh and the octave. The just intervals are summarized in the table below. (For historical reference, note that the diatonic notes match Ptolomy's Diatonic Syntonon.[14])

Note in C Major	**Interval**	**Just Frequency Ratio**	
C	P1	1/1	1.0000
C♯/D♭	m2	16/15	1.0667
D	M2	9/8	1.1250
D♯/E♭	m3	6/5	1.2000
E	M3	5/4	1.2500
F	P4	4/3	1.3333
F♯/G♭	A4/d5/TT	7/5	1.4000
G	P5	3/2	1.5000
G♯/A♭	m6	8/5	1.6000
A	M6	5/3	1.6667
A♯/B♭	m7	16/9	1.7778
B	M7	15/8	1.8750
C	P8	2/1	2.0000

Conversely, then, we can write the harmonics of a note in terms of the just intervals as in the table below.

[14] J. Murray Barbour, *op. cit.*

Harmonic	Ratio	Interval
First (Fundamental)	1/1	unison
Second	2/1	octave
Third	$3/1 = (3/2) \times (2/1)$	octave + fifth
Fourth	$4/1 = (2/1) \times (2/1)$	two octaves
Fifth	$5/1 = (5/4) \times (2/1)^2$	two octaves + major third
Sixth	$6/1 = (3/2) \times (2/1)^2$	two octaves + perfect fifth
Seventh	$7/1 = (7/4) \times (2/1)^2$	two octaves + (septimal) minor seventh
Eighth	$8/1 = (2/1)^3$	three octaves
Ninth	$9/1 = (9/8) \times (2/1)^3$	three octaves + major second
Tenth	$10/1 = (5/4) \times (2/1)^3$	three octaves + major third

Each of the harmonics represents a ratio of a whole number (the harmonic) to 1 (the fundamental). To find the interval, we just have to break it up into parts given in the previous table (mostly, just taking out octaves). Note that the minor seventh doesn't come in so naturally here, it is the septimal minor that ends up in the seventh harmonic (though the inversion comes in as the major second in the ninth harmonic). Important to remember: since the intervals represent ratios, **"adding" or "combining" musical intervals is, mathematically speaking, the same as *multiplying* their ratios together**. So an octave is a ratio of 2/1, and two octaves is a ratio of $(2/1) \times (2/1) = (4/1)$, and an octave "plus" a perfect fifth (3/2) is a total ratio of $(2/1) \times (3/2) = (3/1)$.

2.7.2 How Just is a Just Scale?

Now we have an "ideal" scale in terms of harmonic purity. Unfortunately, this scale is not very practical. While intervals involving the tonic sound tend to be quite consonant, other intervals don't fare so well. For example, the interval from the major second (9/8) to the major sixth (5/3) should also be a perfect fifth (3/2), but it turns out to be flat:

$$\frac{5/3}{9/8} = \frac{40}{27} \approx 1.481. \tag{2.10}$$

The interval connecting the tritone (7/5) to the major seventh (15/8) should be a perfect fourth (4/3), but is sharp:

$$\frac{15/8}{7/5} = \frac{75}{56} \approx 1.339. \tag{2.11}$$

So like the Pythagorean case, the ideal intervals in one key are not so ideal in other keys.

The just scale as we have defined it, has one nice property. The C major diatonic scale contains three **major triads**, or chords comprising three notes sounding together, in particular the tonic, the major third, and the perfect fifth. Clearly, one of these is the **root triad**, denoted by "I," consisting of the notes C-E-G. But the scale also contains the IV triad F-A-C, and the V triad G-B-D, which again contain major thirds and perfect fifths with respect to their relative tonics. These three chords form the chord progression, for example, of "Louie Louie," and about a gazillion other songs in rock, pop, and other styles. The just scale here keeps the major-third and perfect-fifth intervals justly tuned in all three major triads, which works well for songs with these simple progressions.

2.7.3 Temperament

Again, this is a feature of any tuning system, as we already anticipated from Pythagorean tuning, where we required just perfect fifths, but ended up with one wolf fifth and unjust major thirds. Thus comes the art of **musical temperament**, which refers to the compromises involved in the tunings of the notes or intervals of scales, in order to best suit the music to be played. Specifically, **tempering** musical intervals refers to tuning them slightly away from their just tunings, as part of making the overall scale acceptable. There is a distinction between a **tuning** and a **temperament**—both being systems of intervals but the tuning expresses all intervals as rational numbers (as in just intonation or Pythagorean tuning), while a temperament expresses some or all of them as irrationals, due to the compromises involved.[15] The general idea is more or less the same, however, since the compromises involved in temperament, in sacrificing just intervals for another purpose, are inherent in any tuning system.

The Pythagorean tuning, while technically not a temperament, is our first example of the *spirit* of a musical temperament (in compromising the just major thirds in favor of just perfect fifths), and we can compare the Pythagorean tuning to just intonation in the table below.

Note in C Major	Interval	Pythagorean Frequency Ratio		Just Frequency Ratio		Difference from Just
C	P1	1/1	1.0000	(same)		0 ¢
D♭	m2	256/243	1.0535	16/15	1.0667	−21.5 ¢
D	M2	9/8	1.1250	(same)		0 ¢
E♭	m3	32/27	1.1852	6/5	1.2000	−21.5 ¢
E	M3	81/64	1.2656	5/4	1.2500	21.5 ¢
F	P4	4/3	1.3333	(same)		0 ¢
F♯	A4/d5/TT	729/512	1.4238	7/5	1.4000	29.2 ¢
G	P5	3/2	1.5000	(same)		0 ¢
A♭	m6	128/81	1.5802	8/5	1.6000	−21.5 ¢
A	M6	27/16	1.6875	5/3	1.6667	21.5 ¢
B♭	m7	16/9	1.7778	(same)		0 ¢
B	M7	243/128	1.8984	15/8	1.8750	21.5 ¢
C	P8	2/1	2.0000	(same)		0 ¢

The difference between the Pythagorean and just intervals is represented in terms of **cents** (¢). We will soon define mathematically what we mean by cents (Section 3.3, p. 48), but one cent is a small change in the tuning of an interval, and 100 cents is a semitone (give or take, depending on what exactly we mean by a semitone). As we see the Pythagorean tuning pegs the fourths, fifths, major seconds, and minor sevenths, but misses major and minor thirds (not to mention minor seconds and major sevenths) by almost a quarter of a semitone, and misses the tritone by almost a third of a semitone. These differences are significant, as we have mentioned.

2.7.4 Just Tuning

Again, just intonation is not *itself* a temperament, because it insists on "perfect" intervals. The problem, as we see, is that not *all* intervals can be simultaneously just. Thus, we can think of the just scale that we wrote down above (p. 40) as a sort of "just temperament," in the sense of having unjust intervals in other keys.

To hear this, recall that the I, IV, and V triads are consonant in just tuning (where in the C major scale, I=C-E-G, IV=F-A-C, and V=G-B-D). When we play the progression I-IV-V-I, all the chords sound consonant:

But now suppose we "modulate" to C♯, meaning we keep the same tuning for the C scale but think of C♯ as the tonic. Then the triads becomes I=C♯-E♯-G♯, IV=F♯-A♯-C♯, and V=G♯-B♯-D♯. In this case, the IV sounds out of tune, while the I and V are still just.

Finally, modulating to D, we have instead I=D-F♯-A, IV=G-B-D, and V=A-C♯-E). Now the I and V sound bad, while the IV is just.

[15] J. Murray Barbour, *op. cit.*

2.8 Exercises

Problem 2.1

Which of the following intervals are justly tuned in Pythagorean temperament? (All of these are relative to the tonic note C.)

(a) octave

(b) perfect fifth

(c) perfect fourth

(d) major third

(e) minor third

Problem 2.2

Just like 12 fifths approximates 7 octaves, 3 major thirds approximates one octave. The **lesser diesis** (or just **diesis**) is another comma that represents the musical difference (i.e., ratio between) the octave and 3 just major thirds. Compute the diesis as both a ratio and a decimal number.

Problem 2.3

The **schisma** is defined as the musical difference (i.e., ratio between) the Pythagorean comma and the syntonic comma. Compute the schisma as both a ratio and a decimal number.

Chapter 3

Equal Temperament

Arguably the most important temperament is **equal temperament**, which makes an equal compromise around the circle of fifths, as we will explain. This is the temperament represented by the nominal frequencies of the piano keyboard as shown in Section 1.1.4. Equal temperament was well known by the 16th and 17th centuries, but not yet in widespread practical use, except for possibly on fretted string instruments. It became dominant by the end of the eighteenth century.[1] The first calculation of equal temperament in Europe in terms of irrational numbers was by Simon Stevin, c. 1585,[2] though Zhu Zaiyu in China gave the calculation earlier in 1581.[3] Today, it is used almost universally (electronic chromatic tuners, for example, default to equal temperament, referenced to A440), and many musicians don't realize that there are other ways to set the frequencies of musical pitches. However, some argue that much was lost in the widespread adoption of equal temperament.[4]

3.1 Logarithmic vs. Linear Spacing of Pitches

The way we will think about constructing equal temperament is somewhat different than the Pythagorean idea of stacking fifths. Thus, before continuing, we will need to develop the concept of **equal divisions of a musical interval**. To pose a specific problem, suppose we consider the one-octave interval from 100 Hz to 200 Hz, and suppose we want to divide it into equal intervals. One obvious way to do this is to introduce **linear** divisions of the interval: that is, divisions that are of equal width, as measured in Hz. So dividing this into 10 intervals (for 11 total pitches, including the octave), we would obtain 100, 110, 120, . . . , 190, 200 Hz, which is quite a sensible way to divide things up.

But when it comes to *musically* dividing up an interval, this doesn't work out so well. What we want to instead are **logarithmic** divisions of the interval. Before we explain the logarithmic version, compare these two methods of division for yourself. Following are tones starting at middle C (261.63 Hz) and ending one octave higher. The first clip uses linear spacing, and the second logarithmic spacing; but some of the tones are skipped in the pattern of whole tones and semitones in the C major diatonic scale.

The next two clips are again linear vs. logarithmic, but now in the pattern of the chromatic scale (i.e., 12 equal divisions of the octave, where "equal" means something different in each case.

[1]Rudolf Rasch, "Tuning and temperament," in *The Cambridge History of Western Music Theory*, Thomas Christensen, Ed. (Cambridge, 2002), p. 193 (ISBN: 0521686989) (doi: 10.1017/CHOL9780521623711).

[2]Rudolph Rasch, *op. cit.*, pp. 205-7.

[3]Gene J. Cho, *The Discovery of Musical Equal Temperament in China and Europe in the Sixteenth Century*, (Mellen Press, 2003).

[4]Ross W. Duffin, *How Equal Temperament Ruined Harmony (and Why You Should Care)* (Norton, 2006) (ISBN: 0393062279).

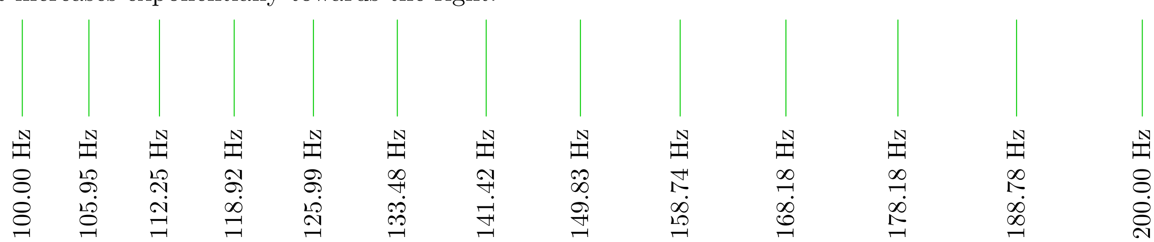
Which scheme of equal spacing sounds more musical? To many, the logarithmic sounds “right,” while the linear spacing sounds like it doesn’t dwell uniformly throughout the octave, rising initially too quickly and dwelling too long at the high end of the octave.

This preference for logarithmic spacing *could* be because we are used to hearing musical notes and scales based on this spacing. However, it is reasonable to believe that the human ear prefers logarithmic spacing by design. Remember that the *intensity* response of the ear is logarithmic, which is why the dB scale is so useful for sound intensities. And recall the layout of the cochlea in the place theory of hearing: different regions along the length of the basilar membrane respond to different frequencies, but roughly the same length of the membrane is allocated to each *octave* (about 3.5 mm). Since octaves represent doubling of frequencies, this is a logarithmic, not a linear spacing, built into the ear.

So how do we divide the interval logarithmically into, say, 12 parts? In terms of linear partitions, we would simply *add* a certain frequency to go from one pitch to the next. Since we are going from 100 to 200 Hz, a range of 100 Hz, the amount we add is 1/12 of 100 Hz, or 8.33 Hz. The resulting partitioning of the octave looks like this:

The locations of the various frequencies are placed linearly across the page, so that with linear divisions of the octave, the divisions have the same amount of (physical) space.

On the other hand, for a logarithmic spacing, instead of *adding* some fixed frequency, we *multiply* by some fixed factor. For 12 steps, this factor turns out to be about 1.05946, as we will deduce in the next section. Plotting this out, we see that the intervals cluster a bit on the low-frequency end, with a spacing that increases exponentially towards the right.



Note that this logarithmic scheme is a bit more “universal” than the linear case. For example, to divide the octave from 200 Hz to 400 Hz into 12 equal parts, for the linear division we would need a *different* additive interval, 1/12 of 200 Hz. But in the logarithmic scheme, we simply multiply by the *same* factor as before.

We have really been building up to the logarithmic spacing of a scale for a while now. Remember that for two pitches to sound consonantly, it is their *ratio* that counts, not the *difference* in frequency. The logarithmic spacing simply fixes the ratio between adjacent pitches (here, the ratio is 1.05946, since that is what we multiply to get from one frequency to the next).

3.2 Twelve-Tone Equal Temperament

Now on to equal temperament, which amounts to dividing up the octave logarithmically, as we just did. The most important case comes from dividing the octave into twelve equal parts, representing the 12 pitches in the chromatic scale. Again, for an equal division, there must be some constant ratio, say R, that relates the frequency of one pitch in the scale to the next one. That is, if f_{n+1} is the frequency of a pitch one semitone higher than f_n, then R is the ratio of these

$$R = \frac{f_{n+1}}{f_n}. \tag{3.1}$$

Solving this for f_{n+1},

$$f_{n+1} = R f_n, \tag{3.2}$$

which means that we always compute the *next* pitch from the *previous* one by multiplying by R.

Now suppose we start with a frequency f_0. The frequency one octave higher is twelve semitones higher, so we can call this f_{12}. Since by the above pattern, $f_1 = Rf_0$, $f_2 = Rf_1 = R^2 f_0$, and so on, we continue this pattern and write

$$f_{12} = R^{12} f_0. \tag{3.3}$$

That is, we took 12 steps (semitones), and so multiplied by R 12 times, to go from f_0 to f_{12}. Since these two frequencies are an octave apart, they are related by a factor of 2, so $f_{12} = 2f_0$. Thus,

$$R^{12} = 2. \tag{3.4}$$

That is, 12 semitones (multiplying by R at total of 12 times) is the same as an octave (multiplying by 2). Now we solve this for R:

$$R = 2^{1/12} = \sqrt[12]{2} \approx 1.059463. \tag{3.5}$$

(equal-temperament semitone)

This is the factor we used in the previous section to divide an octave logarithmically into 12 intervals.

Now, we can go through the whole chromatic scale to compute the frequency ratios of all the musical intervals. For example, the semitone is just R, the whole tone is two semitones, or R^2, the perfect fifth is seven semitones or $R^7 = 2^{7/12}$, and so on. The results are summarized in the table below, with a comparison to the just intervals in cents as before.

Note in C Major	Interval	Equal-Temperament Frequency Ratio		Just Frequency Ratio		Difference from Just
C	P1	$2^{0/12}$	1.0000	(same)		0 ¢
C♯/D♭	m2	$2^{1/12}$	1.0595	16/15	1.0667	−11.7 ¢
D	M2	$2^{2/12}$	1.1225	9/8	1.1250	−3.9 ¢
D♯/E♭	m3	$2^{3/12}$	1.1892	6/5	1.2000	−15.6 ¢
E	M3	$2^{4/12}$	1.2599	5/4	1.2500	13.7 ¢
F	P4	$2^{5/12}$	1.3348	4/3	1.3333	2.0 ¢
F♯/G♭	A4/d5/TT	$2^{6/12}$	1.4142	7/5	1.4000	17.5 ¢
G	P5	$2^{7/12}$	1.4983	3/2	1.5000	−2.0 ¢
G♯/A♭	m6	$2^{8/12}$	1.5874	8/5	1.6000	−13.7 ¢
A	M6	$2^{9/12}$	1.6818	5/3	1.6667	15.6 ¢
A♯/B♭	m7	$2^{10/12}$	1.7818	16/9	1.7778	3.9 ¢
B	M7	$2^{11/12}$	1.8877	15/8	1.8750	11.7 ¢
C	P8	$2^{12/12}$	2.0000	(same)		0 ¢

Note that the fifth (1.4983) is flat compared to the just perfect fifth (1.5), but only slightly, by 2¢.

As a demonstration of this newly tempered fifth, compare the just perfect fifth from before,

to the perfect fifth in equal temperament:

This fifth sounds reasonably good. The major third is however fairly sharp, by 13.7¢, which is significant, though not quite as bad as the Pythagorean major third (sharp by 21.5¢). As a demonstration of this, compare the just major third from before,

to the major third in equal temperament:

The minor third is quite flat, approaching the flatness of the Pythagorean minor third. However, because of the symmetry of the scale, there are advantages of uniformity: the intervals are the same in *every* key. This means, for example, that there is no wolf fifth, because all of the fifths are the same. This also means that an instrument tuned in equal temperament will play equally well (or equally badly) in all keys, which can be a distinct advantage on an instrument like a piano, which is rather cumbersome to retune for different keys.

On the other hand, there have been claims that different keys have different emotional qualities, which can contribute to the "mood" of pieces composed in particular keys. It is difficult to quantify such subjective statements, but one explanation for different qualities of different keys could have come from the use of unequal temperaments in past centuries, where the intervals would have been slightly different depending on the key.[5] Of course, this depends on which temperament was used, but in equal temperament, all keys are effectively equivalent, and no particular emotional character should be assigned to each key.

Most modern musical instruments (pianos, guitars, and so on) are nominally tuned with equal temperament in mind. Other temperaments are easy to achieve on electronic instruments (e.g., keyboard synthesizers). On many instruments, though, such as the violin family, flutes, brass instruments, and reed instruments, there is some flexibility in the nominal pitches via varied fingering or blowing. (Note that this is not the case for the recorder that we mentioned before, because its design constrains the flow of air and thus the pitch for a given fingering.) In this case, any temperament is possible, and for example players who play together and attempt to eliminate beats with respect to other players would play in a sort of "dynamic just intonation," which is perhaps the best possible "temperament."

3.3 Cents

So far, we have been using **cents** to quantify small differences in pitch. For example, most modern electronic tuners show tuning errors expressed in cents. Of course, they can equally well be used to quantify large differences in pitch as well. Once you understand how to set up the standard chromatic scale in equal temperament, is more or less the same thing: a cent is a chromatic semitone (in equal temperament), but divided into 100 equal parts (*logarithmically* equal parts, that is).[6] Thus, chromatic tuners often have a scale ranging from −50 to +50 cents, which is sufficient to display *any* frequency relative to *some* note. You can see this in the photo below of an electronic tuner, where the ends of the range are marked with −50 and +50, relative to the displayed note (in this case, E).

[5] Ross W. Duffin, *op. cit.*

[6] Alexander Ellis originated the concept of cents, and published them in the "Additions by the Translator" in Hermann L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, second English edition translated by Alexander J. Ellis (Dover, 1954), corresponding to the fourth German edition (1877, with notes and additions to 1885) (ISBN: 0486607534). See Appendix XX, p. 430.

One easy way to think about cents is as follows: if there are 100 steps per semitone, then how many cents are there per octave? (How many semitones per octave?) The answer: 1200. So a cent is just like a step in a 1200-note, equally tempered scale. So remember how we handled this before for the chromatic scale. If R is the ratio between successive notes (representing a semitone), and there are 12 semitones per octave, then we solve

$$R^{12} = 2 \tag{3.6}$$

for R, as in Eq. (3.4). In this case, suppose C is the ratio that represents one cent (i.e., the ratio of two frequencies one cent apart). Then there are 1200 cents per octave, so

$$C^{1200} = 2, \tag{3.7}$$

or

$$C = 2^{1/1200} \approx 1.000578. \tag{3.8}$$

So a cent is pretty small, in the sense of representing a ratio that is close to 1.

Now since 100 cents is the same as a semitone (in equal temperament), a perfect fifth is 700 cents, an octave is 1200 cents, and a major third is 400 cents. From the table on p. 47, we can also see, for example, that a perfect-fourth ratio of 4/3 is 2.0 ¢ *flat* of the equal-temperament perfect fourth of 500 ¢. That means that the 4/3 ratio should be the same as 498.0 ¢. As an example, let's express directly this ratio in terms of cents.

Example

Problem. Express the ratio 4/3 in cents.

Solution. Suppose 4/3 corresponds to n cents; then we want to calculate the value of n. Putting this into mathematical form,

$$\frac{4}{3} = C^n, \tag{3.9}$$

where we are expressing n cents as C (one cent) to the nth power—remember, this is a *logarithmic* spacing in frequency.
Now put in the value of $C = 2^{1/1200}$,

$$\frac{4}{3} = 2^{n/1200}, \tag{3.10}$$

and compute the log of both sides:

$$\log\left(\frac{4}{3}\right) = \log\left(2^{n/1200}\right) = \frac{n}{1200}\log 2. \tag{3.11}$$

In the last bit, we used the property $\log x^y = y \log x$ of the logarithm. Now solving for n,

$$n = \frac{1200\log(4/3)}{\log 2} \approx 498.0, \tag{3.12}$$

so 4/3 is the same as 498.0 ¢, just as we read off from the table above.

That is a rather big interval, but cents are particularly useful to represent *small* errors in tuning (i.e., differences in frequency smaller than a semitone). As an example, suppose we want to compute the "error" between the Pythagorean major third and the just major third.

Example

Problem. Express the difference between the the Pythagorean major third (81/64) and the just major third (5/4) in cents.

Solution. The important thing to remember here is that comparing the "difference" between two frequencies means that we want their *ratio*. The ratio of the Pythagorean to the just major third is, as we already computed in Eq. (2.6) as the syntonic comma:

$$\frac{81/64}{5/4} = \frac{81}{80}. \tag{3.13}$$

Now the rest of this problem is to convert the ratio 81/80 to cents. This is just like the previous example, with 4/3 replaced by 81/80. If we go through this calculation again, the result is

$$n = \frac{1200\log(81/80)}{\log 2} \approx 21.5, \tag{3.14}$$

so that the Pythagorean major third is about 21.5 ¢ sharp of the just major third, in agreement with the table on p. 42.

3.3.1 Adding Cents

One point is worth reiterating: **the cent scale is a logarithmic scale for intervals**. What does this mean? When you add the *logarithms* of two numbers, it is the same as the logarithm of the *product*. Mathematically,

$$\log a + \log b = \log ab. \tag{3.15}$$

If a and b are musical intervals (i.e., *ratios*), and we want to *combine* or "add" them, we should *multiply* them. We did exactly this when stacking fifths to make the Pythagorean scale: two stacked fifths was $(3/2)^2$, if we don't care which octave we're in. But in cents, the logarithm takes care of the multiplication, so we really do just *add* the cent values to combine the intervals.

As a quick example. We just showed that a perfect fourth, or 4/3 ratio, is about 498 ¢. Then what is the interval of two perfect fourths? As a ratio, we have to multiply 4/3 by 4/3, or 16/9. In terms of cents, we *add* 498 ¢ to 498 ¢, to get 996 ¢.

3.4 Beyond Standard Western Chromatic and Diatonic Scales

The standard, twelve-note Western chromatic scale has enough complication as we see due to some inherent mathematical imperfections. The diatonic scales are common in Western music, and correspond to choosing seven of the twelve chromatic notes in a particular cyclic pattern of semitones and whole tones. But many other scales are possible and commonly used, and obtained by choosing notes from the chromatic scale in other patterns. An important example is the harmonic minor scale, which includes seven notes but contains an augmented major second (enharmonically equivalent to a minor third) between two of the tones.

However, scales derived from the twelve-note Western chromatic scale are by no means the only possible musical possibilities, and we will review some of the other possibilities in use.

3.4.1 Other Equal Temperaments

In equal temperament, the standard chromatic scale came from dividing the octave into twelve equal steps (semitones), with "equal" used in the logarithmic sense that is standard for pitch intervals. But of course, you could imagine dividing the octave up into any *other* number of such equal intervals. Such scales are called **octave divisions**, and an octave division into N notes or intervals often goes by the shorthand ***N*-EDO** (for N equal divisions of the octave) or ***N*-TET** (for N-tone equal temperament). Therefore, the standard chromatic scale is 12-EDO. Since this is twelve semitones, we could imagine skipping every other semitone to obtain the **whole-tone scale**, which in equal temperament is 6-EDO. This scale pops up in music theory, though is not as commonly used in practice as the diatonic and other scales derived from picking notes out of the chromatic scale. We have also discussed cents as forming an effective 1200-EDO "scale," although pitches one cent apart are generally not distinguishable to the human ear, and so cents are not *directly* useful as a musical scale.

But there are many possible N-EDO scales, so how do we decide which ones are useful? One way to think about this is to remember that the 12-EDO contains all *irrational* ratios with respect to the first note (except, of course, for the unison and octave). But the *nice* sounding intervals are based on simple, *rational* ratios, and so it's not so obvious that 12-EDO should sound nice at all. The reason it works is because of some numerical *coincidences* (see Chapter 8) that cause the 12-EDO intervals to approximately match the rational (justly tuned) intervals. As we discussed, the perfect fifth is a pretty good match (only different by 2 ¢), and the major third is not quite as good (different by 14 ¢), but still better than Pythagorean tuning.

So maybe it's possible to find N-EDO scales with *similar* numerical coincidences, so that at least some of the intervals are consonant by coinciding closely with some nice rational ratios. One approach, for example, is to look for N-EDO scales where the perfect fifth is even *better* than in 12-EDO. This is only possible with finer octave divisions, and it turns out that the good choices of scales that more closely approximate the fifth are 41-, 53-, 306-, and 665-EDO (see Section 8.3.3), which have fifths that are +0.48 ¢ (for the 24th note), −0.068 ¢ (for the 31st note), +0.0058 ¢ (for the 179th note), and −0.000 11 ¢ (for the 389th note) mistuned from the just ratio (3/2). Of course, it is possible to continue to higher accuracy, but 306-EDO is already pretty unreasonable, though 53-EDO is relatively common, at least in theoretical discussions. The improved tunings possible in high-N-EDO give one motivation for **microtonal music**, which employs intervals smaller than the 12-EDO half-step interval.

Another often-discussed choice is 19-EDO.[7] While this doesn't improve upon the perfect fifth of 12-EDO, its fifth is still broadly feasible (−7.2 ¢ of just), and it manages to coincide well with major and minor thirds, the major second, the fourth, and all the inversions, for a total of 8 consonances other than the unison and octave.[8] Also, by adapting standard enharmonic musical notation by allowing enharmonic notes (e.g., C♯ and D♭) to represent different *chromatic* tones (instead of being different by only a comma as in standard enharmonic usage) in the scale, and also using the two notes "E♯/F♭" and "B♯/C♭," it is possible to represent all of the notes in 19-EDO. All this is summarized in the table below.

Note in C Major	Interval	Equal-Temperament Frequency Ratio		Just Frequency Ratio		Difference from Just
C	P1	$2^{0/19}$	1.0000	(same)		0 ¢
C♯		$2^{1/19}$	1.0372			
D♭		$2^{2/19}$	1.0757			
D	M2	$2^{3/19}$	1.1157	10/9	1.1111	7.1 ¢
D♯		$2^{4/19}$	1.1571			
E♭	m3	$2^{5/19}$	1.2001	6/5	1.2000	0.1 ¢
E	M3	$2^{6/19}$	1.2447	5/4	1.2500	−7.4 ¢
E♯/F♭		$2^{7/19}$	1.2909			
F	P4	$2^{8/19}$	1.3389	4/3	1.3333	7.2 ¢
F♯		$2^{9/19}$	1.3887			
G♭		$2^{10/19}$	1.4402			
G	P5	$2^{11/19}$	1.4938	3/2	1.5000	−7.2 ¢
G♯		$2^{12/19}$	1.5493			
A♭	m6	$2^{13/19}$	1.6068	8/5	1.6000	7.4 ¢
A	M6	$2^{14/19}$	1.6665	5/3	1.6667	−0.1 ¢
A♯		$2^{15/19}$	1.7284			
B♭	m7	$2^{16/19}$	1.7927	9/5	1.8000	−7.1 ¢
B		$2^{17/19}$	1.8593			
B♯/C♭		$2^{18/19}$	1.9284			
C	P8	$2^{19/19}$	2.0000	(same)		0 ¢

Additionally, 31-EDO and 34-EDO are relatively important as having good matches to the thirds and fifth, as well as to other rational ratios.

In modern music, dissonant intervals are much more acceptable to the ear than they were centuries ago, and it is not strictly necessary to use scales with many consonances. For example, the composer Easley Blackwood has an album of compositions in every equal temperament ranging from 13-EDO to 24-EDO, played on synthesizer (including several compositions played by Jeffrey Kust on a special guitar in 15-EDO).[9] Though at first listening, this music can sound a lot like music played on instruments in bad need of tuning.

3.4.2 Non-Western Practice

Non-12-EDO scales are important outside of the west. For example, Arabic music (in the *maqam* system) employs scales with **microtones**—intervals smaller than semitones in 12-EDO. Simplistically, the scales can be thought of as pitches selected from 24-EDO, which is composed of "quarter tones." This comparison allows for adaptation to Western pitch notation, where the quarter tones in between the 12-EDO semitones

[7] Mayer Joel Mandelbaum, *Multiple Division of the Octave and the Tonal Resources of 19-tone Temperament*, Ph.D. dissertation, Indiana University (1961). Available online at `http://anaphoria.com/mandelbaum.html`.

[8] Mandelbaum, *op. cit.*, p. 385.

[9] Easley Blackwood, *Microtonal* (Cedille, 1994). See `https://www.youtube.com/watch?v=YJQsR-Z5aDc`.

are notated as half-flatted pitches (represented by a flat symbol with a bar). These quarter tones are used, for example, on the oud (a fretless string instrument);[10] without preset pitches, the instrument easily accommodates the microtones.

One interesting example of 5-EDO practice (a pentatonic scale, but different from the Pythagorean pentatonic scale we discussed in Section 2.2) occurs in Uganda. The **akadinda** of the Baganda people in Buganda (a kingdom within Uganda) is a 17-key, xylophone-like instrument, with wooden keys fixed with twigs to two banana-tree-stalk supports. The keys are tuned close to 5-EDO.[11] Since the "chromatic" step in 5-EDO is 240 ¢, these notes fall well in between the usual Western chromatic pitches. This gives the percussion an atonal quality. The akadinda is typically played in a rapid, repeating pattern by three players.[12,13] Also, in gamelan music of Java, there are two scale systems: **slendro** and **pelog**. Although there is variation, slendro tuning is close to 5-EDO, though the intervals vary from about 230 to 250 ¢ (exact 5-EDO is 240 ¢).[14,15]

In the classical music of India, the scale is based on intervals called **shrutis** (transliterated in various ways, e.g., **sruthis**). Even the number of shrutis in an octave has been the subject of debate, but a common modern number is 22, and this number is supported by analysis of vocal recordings in North-Indian (Hindustani) music.[16,17] A general complication in classifying pitches in Indian classical music is that the pitches themselves are not fixed—Indian musicians value the freedom of varied intonation.[18] It can be argued that the 22 shrutis in South-Indian (Carnatic) music are inflections of 12 "basic" pitches that more or less coincide with the Western chromatic scale.[19]

3.4.3 Western Practice

In Western music, we have already noted some deviation from 12-EDO in terms of both temperament and other octave divisions. We have also noted the broad use of pentatonic scales in Section 2.2. However, pentatonic scales are essentially simplifications of standard diatonic scales, so aren't really very different.

One common example of a non-chromatic note is the **blue note** in American blues music. The prevailing style adds interest and tension by juxtaposing minor and major tones together in the same melodic phrase, especially minor and major thirds. It is also common to play pitches in between the minor and major third. On guitar, this is easy to do by bending the minor third up slightly, roughly by a quarter tone, or even just by bending slowly from the minor to the major third to pass through the blue note. ("Bending" a string on a guitar involves pulling a fretted string to one side to raise its pitch.) On piano, where the in-between pitches are technically impossible, it is common to emulate this note, for example, by playing the minor third as a grace note to the major third. The rapid passage between the two notes acts as the

[10]For a good musical example on the oud, see `http://www.youtube.com/watch?v=6R7JZbydqVk` for a recording by Rahim Al-Haj, The microtones here add some nice musical tension. At a concert, Al-Haj related a story about a woman talking to him after a prior concert, who said that she enjoyed his playing, but wondered why he kept playing out of tune.

[11]James T. Koetting, "Africa/Ghana," in *Worlds of Music*, 2nd ed., Jeff Todd Titon, Ed. (Schirmer, 1992), pp. 94-6 (ISBN: 0028726022).

[12]A good example of an akadinda trio in action: `http://www.youtube.com/watch?v=gJzWOC--ixc`.

[13]For more playing examples and details of the construction, see `http://www.youtube.com/watch?v=5u28FXJv2K4`. This is a Dutch-language film, with interviews in English.

[14]Wasisto Surjodiningrat, P. J. Sudarjana, and Adhi Susanto, *Tone Measurements of Outstandind Javanese Gamelan in Yogyakarta and Surakarta*, 2nd rev. ed. (Gadjah Mada University Press, 1993) (ISBN: 9794202738).

[15]For a tuning demonstration, see `https://www.youtube.com/watch?v=3Ku9iH2pU9g`.

[16]A. K. Datta, R. Sengupta, and N. Dey, "Objective Analysis of Shrutis from the Vocal Performances of Hindustani Music Using Clustering Algorithm," *Eunomios*, 28 June 2011, (non-refereed paper) available at `http://www.eunomios.org/contrib/datta-dey-sengupta2/datta-dey-sengupta2.pdf`.

[17]Note that the shruti is distinct from the **svara**: there are 7 unequally spaced svaras in the octave, and these are the scale degrees. The relation between shrutis and svaras is complicated, but roughly speaking, the svara intervals can be described in terms of 2, 3, or 4 shrutis. See Suvarnalata Rao and Wim van der Meer, "The Construction, Reconstruction, and Deconstruction of Shruti," in *Hindustani Music: Thirteenth to Twentieth Centuries*, Joep Bor, François Nalini Delvoye, Jane Harvey, and Emmie te Nijenhuis, Eds (Manohar, 2010), p. 673 (ISBN: 8173047588). Available online at `http://dare.uva.nl/document/2/76993`.

[18]Bonnie C. Wade, *Music in India: The Classical Traditions* (Manohar Publishers, 2008), pp. 31-32.

[19]Arvindh Krishnaswamy, "Pitch Measurements versus Perception of South Indian Classical Music," *Proceedings of the Stockholm Music Acoustics Conference, August 6-9, 2003 (SMAC 03), Stockholm, Sweden* (2003). Available online at `https://ccrma.stanford.edu/~arvindh/cmt/smac03.pdf`.

in-between pitch.

3.5 Exercises

Problem 3.1

(a) If you invented a "septatonic" scale with 7 equally spaced notes per octave (in the same sense as the equal-temperament chromatic scale, which would be "dodecatonic" with 12 equally spaced steps), what would be the frequency ratio between adjacent notes? In other words, what is the ratio between adjacent notes in **7-EDO**? *Explain.*

(b) Suppose you set the tonic (first note in the scale) at exactly 400 Hz. What are the frequencies of the next two notes?

Problem 3.2

(a) What is the ratio between the frequencies 200.00 Hz and 220.00 Hz? Express the ratio as a fraction *and* as a decimal number.

(b) What is the same ratio in (a), expressed in cents?

(c) Suppose that these two frequencies are pitches *two* steps apart on some equally spaced musical scale. (Equally spaced here is in the musical sense!) What is the ratio between 200.00 Hz and the *next* pitch up from 200.00 Hz in this scale?

(d) What is the frequency of the in-between pitch that you calculated in (c)? Give your answer using an *appropriate* number of decimal places.

(e) What is the ratio in (c), expressed in cents?

Problem 3.3

Suppose a piano string is *supposed* to be tuned to 440 Hz, but in reality it is tuned to 441 Hz. How far out of tune is it, expressed in cents?

Problem 3.4

Remember that a scale with N equally spaced notes per octave is called N-EDO, for N "equal divisions of the octave." That is, the usual equal temperament is 12-EDO.

(a) Suppose that in some N-EDO scale, the frequencies of two adjacent notes in the scale are 440.00 Hz and 466.16 Hz. What is N?

Hint: set this up and solve the equation using logarithms. Remember that $\log a^b = b \log a$. Or maybe you can find a simpler way to solve this problem.

(b) What is the interval between two adjacent notes in 12-EDO, expressed in cents? (That is, what is a minor second in equal temperement, expressed in cents?)

(c) What is the interval between two adjacent notes in 24-EDO, expressed in cents? (That is, what is a "quarter step" in equal temperement, expressed in cents?)

Problem 3.5

(a) The **just-noticeable difference** (JND) in frequency for human hearing is the smallest frequency changed between two successively played pure tones that the average person can distinguish. If the JND is about 0.5% around 440 Hz, what is the JND in cents? (You may round to the nearest cent.)

(b) What is the difference (actually, the ratio) between the equal-temperament fifth and the just perfect fifth, expressed in cents? (You may round to the nearest cent.)

Problem 3.6

Compute the diesis (Problem 2.2) in cents.

Problem 3.7

(a) Compute the Pythagorean comma in cents.

(b) Compute the syntonic comma in cents.

(c) Compute the schisma (Problem 2.3) in cents.

(d) The Pythagorean and syntonic commas are said to be close to 12 and 11 times the schisma, respectively. This means, for example, the syntonic comma is approximately 11 schismas. Calculate these ratios more precisely, to at least a few more decimal places.

Problem 3.8

Suppose we want to define a *finer* unit for musical intervals than the cent. Let's define the **mil** such that the equal-temperament minor second (half-step interval) is divided equally into 1000 mils. ("Equally" here in the same sense as cents.)

(a) What is the ratio of two frequencies 1 cent apart?

(b) How many mils per cent?

Problem 3.9

One popular method of tuning two adjacent guitar strings (or bass-guitar strings) is as follows.

1. Play the *fourth* harmonic of the *lower* string, by touching it lightly at the fifth fret (1/4 of the way from the "nut end" of the string) and plucking it.
2. Play the *third* harmonic of the *higher* string, by touching it lightly at the seventh fret (1/3 of the way from the "nut end" of the string) and plucking it.
3. Listen for beats, and adjust the tension of one of the strings to eliminate any beating.

(a) *Assume* the two strings are *supposed* to be tuned to a just fourth. (What is the ratio for a just fourth? Read the temperament notes!) Argue that the tuning procedure gives the correct tuning. (Note in particular that this procedure "works" for all adjacent pairs of strings on the standard guitar *except* for one, the G-B pair, which is tuned to a major third.)

(b) Now realize that any standard guitar is designed to be tuned in equal temperament, so the strings, should be tuned to an *equal-temperament* fourth, not a just fourth. Thus argue that the above tuning technique does *not* tune the two strings correctly.

(c) Based on your answer to (b), by tuning via this method, by how much are the two strings out of tune? Express your answer as a ratio and in cents.

Problem 3.10

Write down decimal values for all the interval ratios in 7-EDO. Which of these intervals are relatively close to intervals in the just scale (i.e., which ones are close to simple fractions)?

Chapter 4

Meantone Temperaments

Remember that Pythagorean tuning has pure fifths (except for one), but unpleasantly sharp thirds. This was not a problem, for example, for early music such as Gregorian chants (formalized around the time of the papacy of Pope Gregory I, AD 590–604), where groups sing only in unison, or in **monophony**. But **polyphony**—the harmony of multiple pitches—slowly developed. Its origins are in the development of **organum**, or the extension of monophonic chants to two parallel voices, where one voice is transposed to a fixed, consonant interval (which meant fifth, fourth, or octave at the time) above the other. Here again, Pythagorean tuning suits this music, since it has pure fifths and fourths. The parallel organum later developed into true polyphony, for example with a melody over a droning line, or contrary motion of lines. In this more sophisticated polyphony, other intervals besides the fourth and fifth become important. One important interval is the major third, which we have described as a consonant interval, at least in the just-intonation ratio of 5/4. In early music, the third was in fact regarded as a dissonant interval; its consonant status came only later. The increasing use of thirds (major and minor) around the time of the early Renaissance motivated the theory and use of other tuning systems besides Pythagorean.[1] Of course, this is mainly an issue for musical instruments with fixed tuning. In vocal music or instruments with easily adjusted intonation (e.g., wind instruments or fretless string instruments), players can dynamically adjust to whatever "sounds best."

The modern solution to tuning is equal temperament, by default. Equal temperament was known by the 16th century, and was probably the norm on fretted string instruments (lute, vihuela, viol or viola da gamba, and later guitar) from the mid or early 16th century, because of the constraints on tuning from having multiple strings share common frets.[2] However, equal temperament was *not* the standard on keyboard instruments, which were not so constrainted, until around the end of the 18th century. In fact, on string keyboard instruments, where the tuning could be tweaked relatively quickly (compared to the pipe organ), performers probably retuned between pieces to accommodate different keys.[3] Also, remember that equal temperament improved the thirds somewhat: the major third is only 13.7¢ from just, compared to the Pythagorean error of 21.5 ¢, and the minor third is also improved to −15.6 ¢ from the Pythagorean −21.5 ¢. However, the equal-temperament thirds still aren't *that* good.

One important approach to improving the purity of thirds comes in the form of **meantone temperaments**. The idea here is fairly simple: chain perfect fifths together to form the chromatic scale, just as in Pythagorean tuning. The main difference is to not use the *just* perfect fifth (3/2). Since the Pythagorean tuning overshot the major third, the most benefit comes by choosing a slightly *flat* perfect fifth. Thus, meantone temperaments obtain better-tuned thirds at the *expense* of the perfect fifths.

[1]Catherine Nolan, "Music theory and mathematics," in *The Cambridge History of Western Music Theory*, Thomas Christensen, Ed. (Cambridge, 2002), p. 272 (see p. 276 in particular) (ISBN: 0521686989) (doi: 10.1017/CHOL9780521623711).

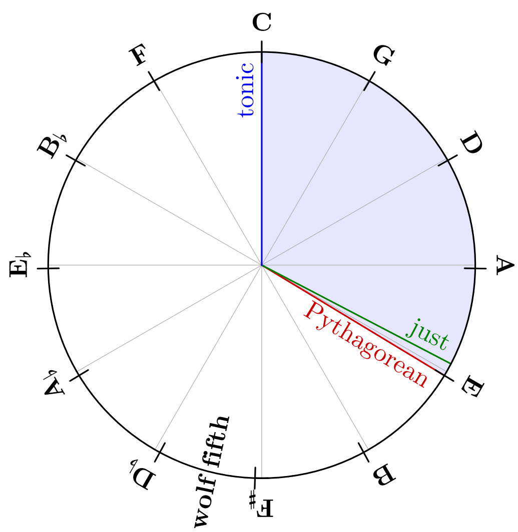
[2]J. Murray Barbour, *Tuning and Temperament: A Historical Survey*, 2nd ed. (Michigan State College, 1953) (ISBN: 0486434060), p. 188. First edition available online at `https://archive.org/details/tuningtemperamen00barb`. See also Mark Lindley, *Lutes, Viols, and Temperaments* (Cambridge, 1984) (ISBN: 0521288835).

[3]J. Murray Barbour, *op. cit.*, p. 191.

4.1 Quarter-Comma Meantone Temperament

The most important meantone temperament, at least in a theoretical sense, is **quarter-comma meantone temperament**. Indeed, "meantone temperament" refers to this temperament unless otherwise specified. The goal in quarter-comma meantone is to tweak the fifths in order to obtain *pure* major thirds (in a ratio of 5/4). The first description of meantone is attributed to Piero Aron (Pietro Aaron) in 1523.[4] As Barbour wrote: "Thus arose the system that, with various modifications, was to be the strongest opponent of equal temperament, so far as keyboard instruments were concerned, for two or three hundred years."[5] Meantone was probably widely used on keyboard instruments from the late 16th century into the 18th century.[6] Meantone tuning is still used in performances of classical music, particularly on harpsichord.[7] Although not particularly common, some historic and historically informed pipe organs are also tuned to meantone.[8] For example, the Brombaugh "Opus 25" organ (1981) at Oberlin college in Oberlin, Ohio is in quarter-comma meantone.[9]

Recall that in Pythagorean tuning, we stacked four just fifths to get the major third. That is, on the circle of fifths, the major third (E) is at the fourth position, or four fifths, along the clockwise direction from the tonic (C). Also, recall that a major disadvantage of Pythagorean tuning is that the major third is sharp by the syntonic comma, 81/80. This situation is shown in the *Pythagorean* circle of fifths below, with the just major third drawn in (the gap between the major thirds is exaggerated to make it more visible).



So the idea behind quarter-comma meantone is to "squish" *all* the fifths to improve the major third. More precisely, the idea is to squish the fifths just so that the four stacked fifths connecting the tonic to the major third (the shaded region in the diagram) gives *exactly* the pure major third (5/4). Another way to think about this is that we want to distribute the syntonic comma over the chain of four fifths—hence the *quarter comma*. (The "meantone" comes from the Pythagorean construction, which yields major thirds that are exactly two whole tones, so that the whole tone is the *geometric mean* of the tonic and major third. That is, the major-second interval is the square root of the major-third interval.) The quarter-comma correction

[4]Piero Aron, *Toscanello in musica* (1523). Revised edition from 1529 available online at `https://archive.org/details/toscanelloinmvsi00aaro`. See citation and info on usage in Rudolf Rasch, "Tuning and temperament," in *The Cambridge History of Western Music Theory*, Thomas Christensen, Ed. (Cambridge, 2002), p. 193 (ISBN: 0521686989) (doi: 10.1017/CHOL9780521623711).

[5]J. Murray Barbour, *op. cit.*, p. 10.

[6]Rudolf Rasch, *op. cit.*, p. 202.

[7]Example: Kerll's Toccata 1, Natascha Reich on harpsichord, `http://www.youtube.com/watch?v=2--_nMgLmVU`.

[8]For a list in North America, see `http://en.wikipedia.org/wiki/Meantone_organs_in_North_America`.

[9]Homer Ashton Ferguson III, *John Brombaugh: The Development of Americas Master Organ Builder*, D.M.A. dissertation (2008), p 151.

is $\sqrt[4]{80/81}$, which taken four times gives

$$\left(\sqrt[4]{\frac{80}{81}}\right)^4 = \frac{80}{81}, \tag{4.1}$$

which exactly cancels the syntonic comma 81/80. Thus, the corrected perfect fifth is the just perfect fifth times the quarter-comma correction, or

$$\frac{3}{2}\sqrt[4]{\frac{80}{81}} = \sqrt[4]{5} = 5^{1/4}. \tag{4.2}$$

(quarter-comma meantone fifth)

We can see that this directly works out, since four of these perfect fifths is a ratio of 5/1, which reduced by two octaves becomes the just major third 5/4.

Repeating the Pythagorean construction, by starting at the tonic C, and proceeding six fifths clockwise to F♯ and five fourths counterclockwise to D♭, we obtain the temperament in the table below, which we again compare to just intonation.

Note in C Major	**Interval**	**Meantone Frequency Ratio**		**Just Frequency Ratio**		**Difference from Just**
C	P1	1/1	1.0000	(same)		0 ¢
D♭	m2	$2^3/5^{5/4}$	1.0700	16/15	1.0667	5.4 ¢
D	M2	$5^{2/4}/2^1$	1.1180	9/8	1.1250	−10.8 ¢
E♭	m3	$2^2/5^{3/4}$	1.1963	6/5	1.2000	−5.4 ¢
E	M3	5/4	1.2500	(same)		0 ¢
F	P4	$2^1/5^{1/4}$	1.3375	4/3	1.3333	5.4 ¢
F♯	A4/d5/TT	$5^{6/4}/2^3$	1.3975	7/5	1.4000	−3.0 ¢
G	P5	$5^{1/4}$	1.4953	3/2	1.5000	−5.4 ¢
A♭	m6	8/5	1.6000	(same)		0 ¢
A	M6	$5^{3/4}/2^1$	1.6719	5/3	1.6667	5.4 ¢
B♭	m7	$2^2/5^{2/4}$	1.7889	16/9	1.7778	10.8 ¢
B	M7	$5^{5/4}/2^2$	1.8692	15/8	1.8750	−5.4 ¢
C	P8	2/1	2.0000	(same)		0 ¢

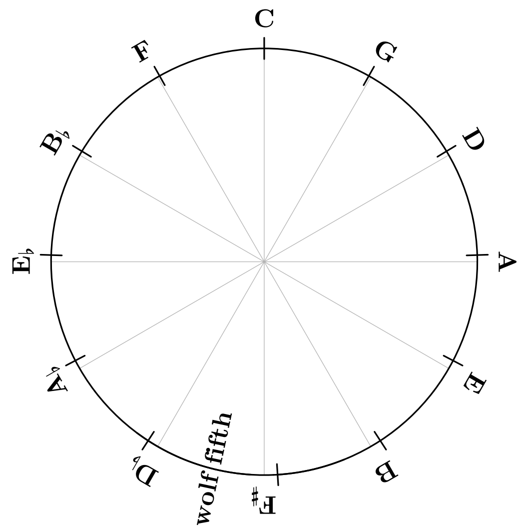
Note that the perfect fourth and fifth are out of tune by about five cents, but this is not nearly as bad as the Pythagorean major third. The tuning of the minor third and major sixth are also much better than their Pythagorean counterparts. Indeed, the worst intervals are the (dissonant) major second and minor seventh, but even they are improved.

However, the wolf fifth still remains: the interval from F♯ ($5^{6/4}/2^3$) to D♭ ($2^3/5^{5/4}$), or double this for the next octave) is

$$R_{\text{wolf}} = \frac{2^7}{5^{11/4}} \approx 1.5312, \tag{4.3}$$

(meantone wolf fifth)

or about 35.7 ¢ sharp (over a third of a semitone!). This is even worse than the Pythagorean wolf fifth, which was 23.5 ¢ flat. This wolf fifth is illustrated below (again, the size of the wolf fifth is exaggerated for dramatic effect).



Thus, by flattening the fifths to obtain just major thirds, we sort of went overboard in terms of the final gap on the last perfect fifth. Indeed, the term "wolf fifth" originally applied to the wolf fifth in quarter-comma meantone tuning, though the "wolf" term now more generally applies to any interval that is unpleasantly out of tune.

To hear the wolf fifth here, compare again two asymmetric-triangle-wave tones sounding a *just* perfect fifth,

with a quarter-comma meantone wolf fifth:

The latter is unpleasantly sharp. Thus, all the considerations we mentioned for the Pythagorean wolf fifth apply here: while the intervals within the diatonic scale of the tonic are better tempered, and other intervals in related keys could work, it was important to avoid the particularly bad wolf interval, either with more complicated tunings, or through the composition.

Note also that while most of the major thirds are just (eight of them, in fact), the remaining four major thirds (E-A♭, F♯-B♭. A-D♭, and B-E♭, in our construction) are distinctly sharp, forming the interval $32/25 = 1.28$, about 41.1 ¢ sharp, and these intervals are also best avoided.

Again, to hear this, compare the just ratio of 5/4, which sounds smooth,

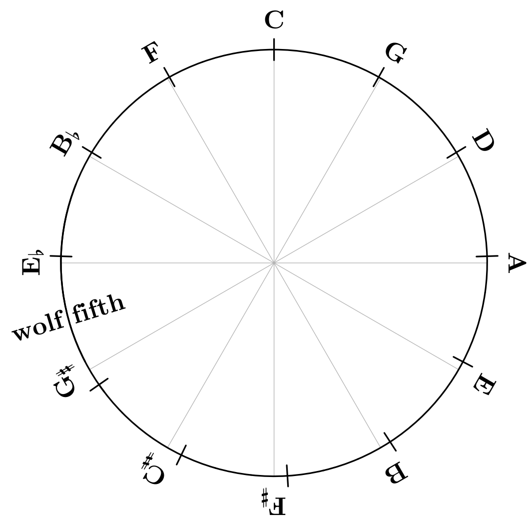
to the badly out-of-tune 32/25 major third:

Despite its shortcomings, quarter-comma meantone tuning was in popular use during the Renaissance, until around the end of the seventeenth century. As noted above, one of the consequences of the construction of this temperament is the "spiral" of fifths, so that flatted notes are slightly sharper than the enharmonically equivalent sharps (i.e., A♭ is distinct from and slightly sharper than G♯). There are numerous references to these slight differences in pitch in the musical literature, particularly for performance of fretless string instruments like violin and cello,[10] where it is natural to express such fine differences in pitch for stopped notes.

[10] Ross W. Duffin, *How Equal Temperament Ruined Harmony (and Why You Should Care)* (Norton, 2006) (ISBN: 0393062279).

4.2 "Conventional" Quarter-Comma Meantone

The way we described meantone temperament was by stacking fifths in exactly the same way as in Pythagorean tuning. However, the common usage of meantone temperament is somewhat different: basically it was constructed so the wolf fifth is between G♯ and E♭, as shown below (as usual, with exaggerated commas).



This is essentially the same idea, but taking D as the root note instead of C. The reason is that on C instruments (i.e., keyboard instruments), music tends to reside in keys near C around the cycle of fifths (that is, keys with not too many flats or sharps). Then, for example, major chords in those keys tend to involve notes in the *clockwise* direction from the root note of the chord on the circle of fifths (recall that the fifth is one "notch" counterclockwise, and the third is four "notches" counterclockwise), so it turns out to be more convenient to hide the wolf just before the nine o'clock position. Essentially, the "spirit" of meantone is to prioritize pure major thirds, and this wolf position tends to maximize the pure thirds available.

The intervals for conventional meantone are listed below. The two modified notes are highlighted. Note that they are quite far away from the corresponding just intervals.

Note in C Major	Interval	Meantone Frequency Ratio		Just Frequency Ratio		Difference from Just
C	P1	1/1	1.0000	(same)		0 ¢
C♯	m2	$5^{7/4}/2^4$	1.0449	16/15	1.0667	−35.7 ¢
D	M2	$5^{2/4}/2^1$	1.1180	9/8	1.1250	−10.8 ¢
E♭	m3	$2^2/5^{3/4}$	1.1963	6/5	1.2000	−5.4 ¢
E	M3	5/4	1.2500	(same)		0 ¢
F	P4	$2^1/5^{1/4}$	1.3375	4/3	1.3333	5.4 ¢
F♯	A4/d5/TT	$5^{6/4}/2^3$	1.3975	7/5	1.4000	−3.0 ¢
G	P5	$5^{1/4}$	1.4953	3/2	1.5000	−5.4 ¢
G♯	m6	$5^2/2^4$	1.5625	8/5	1.6000	−41.1 ¢
A	M6	$5^{3/4}/2^1$	1.6719	5/3	1.6667	5.4 ¢
B♭	m7	$2^2/5^{2/4}$	1.7889	16/9	1.7778	10.8 ¢
B	M7	$5^{5/4}/2^2$	1.8692	15/8	1.8750	−5.4 ¢
C	P8	2/1	2.0000	(same)		0 ¢

4.3 Other Meantone Temperaments

Other meantone temperaments, with less of a correction than the quarter-comma (i.e., fifth-comma or sixth-comma, where the comma is still the syntonic comma) were also historically important. These were even more of a compromise, not giving any perfect thirds or fifths, but also reducing the extreme mis-tunings of the worst intervals. Fifth-comma meantone, incidentally, has the interesting property (Problem 4.4) of exactly balancing the compromise between mistuning the thirds and the fifths (at least the ones that don't cross the wolf interval). Sixth-comma meantone is more uniform around the circle of fifths, and matches the popular Vallotti temperament (Section 5.5) on half of the fifths (the other half are pure in Vallotti). Both fifth- and sixth-comma meantone enjoyed common historic use, sixth-comma in particular for music requiring more modulation than is feasible with quarter-comma meantone.[11]

Another interesting possibility is to focus on the *minor* thirds instead of the major thirds. Remember that on the circle of fifths, the *major* third (E) was four fifths clockwise from the tonic C. By contrast, the *minor* third (E♭) is *three* fifths *counter*clockwise from C. If we then insist on pure minor thirds in a meantone temperament, we then obtain **third-comma meantone**, because the syntonic comma should be distributed over the three fifths between E♭ and C.

4.4 Equal Temperament as a Meantone Temperament

Returning to equal temperament from Chapter 3, it is useful to realize that it is in some sense another special case of a meantone temperament: the idea being the Pythagorean comma ($3^{12}/2^{19}$) must be distributed in twelve equal parts, narrowing (tempering) each of the fifths. Then "1/12" of the Pythagorean comma is $\sqrt[12]{3^{12}/2^{19}}$, and tempering the just perfect fifth (3/2) means that we divide by this correction:

$$\frac{3/2}{\sqrt[12]{3^{12}/2^{19}}} = \frac{3}{2}\sqrt[12]{\frac{2^{19}}{3^{12}}} = \sqrt[12]{2^7} = 2^{7/12}. \tag{4.4}$$

This is exactly the equal-tempered fifth. Note, however, that the meantone temperaments are named according to the fraction of the *syntonic* comma, not the *Pythagorean* comma—so equal temperament is *not* twelfth-comma meantone. Because the syntonic comma is about 11/12 times the Pythagorean comma (see Problem 2.2), equal temperament turns out to be more like eleventh-comma meantone (see Problem 4.5).

Again, note that the equal-temperament fifth ($2^{7/12} \approx 1.4983$) is slightly flat of the just perfect fifth ($3/2 = 1.5$). However, the difference is slight, only 2 ¢, a smaller difference than in quarter-comma meantone tuning (5.4 ¢).

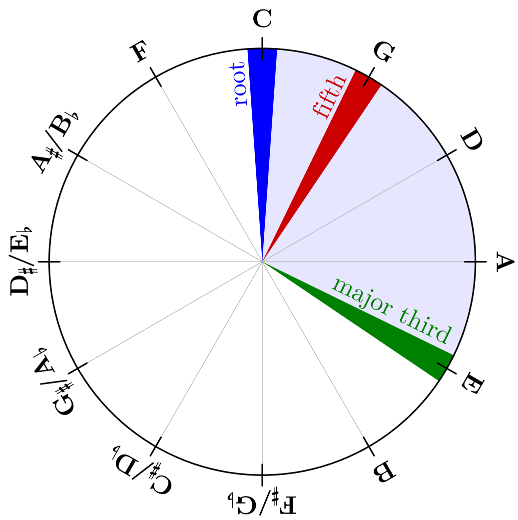
4.5 Listening Examples: All Major Triads

Now that we have accumulated a few different temperaments, it is important to *listen* to them in a way that makes their differences obvious. Small adjustments in tuning can seem like a subtle business. But comparing things with small differences side-by-side is the most efficient way to understand an appreciate the differences—to become a "connoisseur" of temperaments. The same thing applies to comparing good wines, or coffees, or teas, or chocolates. In tasting good but subtly different examples on different days, it is difficult to gain an appreciation for the differences or even to notice the differences—differences that are obvious in a side-by-side comparison.[12]

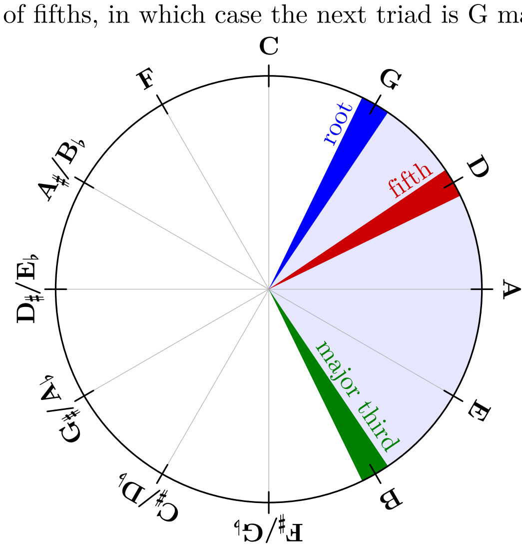
As the example, we will listen to all the major triads. Here, the C-major triad is the *root* C, the major third (E), and the perfect fifth (G). On the circle of fifths, these three notes define a "wedge" of 1/3 of the circle, as shown below.

[11] J. Murray Barbour, *op. cit.*, pp. 35 and 43.

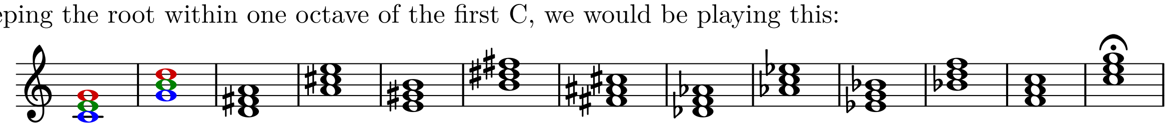
[12] Seth Roberts noted that side-by-side comparisons lead to appreciation and connoisseurship called this the **Willat effect**: `http://blog.sethroberts.net/2011/07/08/the-willat-effect-side-by-side-comparisons-create-connoisseurs/`.



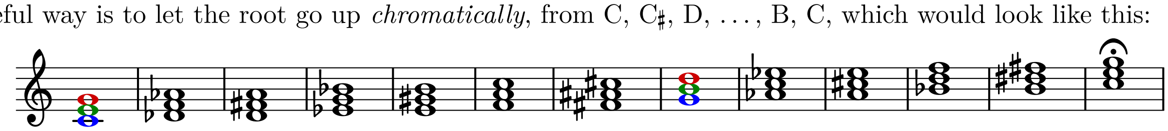
Then we can proceed the the next triad, with *next* possibly meaning different things. One obvious choice is just to move around the circle of fifths, in which case the next triad is G major (G-B-D).



And we can keep going around until we get back to C, with the sequence C, G, D, . . . , B♭, F, C. Musically, keeping the root within one octave of the first C, we would be playing this:



(Colors in the first two triads match the two circles of fifths above.) Note that in this section, we aren't bothering to make distinctions between enharmonic notes (e.g., A♯/B♭). On the other hand, another musically useful way is to let the root go up *chromatically*, from C, C♯, D, . . . , B, C, which would look like this:



We will do *both*: the cycling by fifths makes it easier to understand when strong dissonances come up, while the chromatic cycling makes it a bit easier to connect to actual music.

4.5.1 Equal Temperament

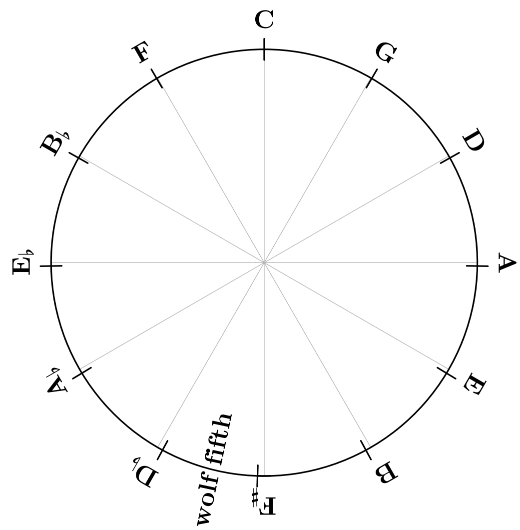
The first example is equal temperament, which is fairly simple. Note that the temperament is the same for all major triads, the price being that all of them are somewhat out of tune (the thirds more than the fifths).

(by fifths) (chromatic)

In the chromatic example, see if you can hear the beating increase in frequency as the triads progress.

4.5.2 Pythagorean Tuning

Now listen to Pythagorean tuning. Notice that most of the triads should have similar tuning except when the triad overlaps the wolf-fifth interval. The “typical” triad sounds fairly out of tune because the major third is so sharp (Pythagorean tuning is of course *great* for harmonies *in fourths and fifths*). Note that the A, E, and B triads sound *better*: they overlap the wolf fifth, which tempers the sharpness of the major third, without changing the fifth. But the F♯ triad sounds worse because now the fifth really is the wolf fifth.



(by fifths) (chromatic)

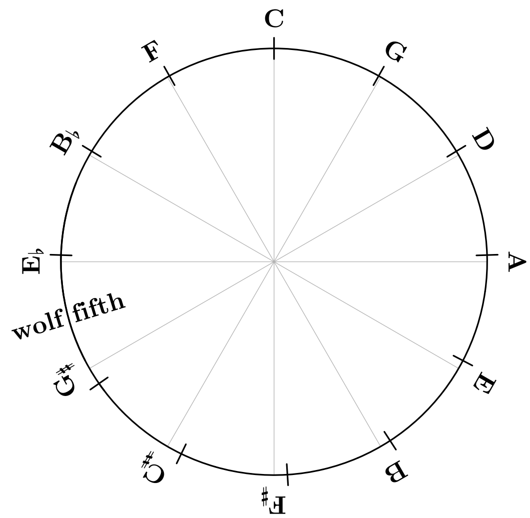
4.5.3 Just Temperament

Remember that just tuning keeps the C, F, and G triads just, at the expense of all the others. This is pretty obvious when you listen to it. Some are pretty horrendous, though note that D♭ and A♭ are also just and thus sound pretty nice.

(by fifths) (chromatic)

4.5.4 Meantone Temperament

Remember that in (quarter-comma) meantone, the fifths are slightly flat, but the thirds are pure. Except, of course, when the triads cross the wolf fifth, which is conventionally between G♯ and E♭.



Most of the triads have a “sweet” sound, with the slightly flat fifth giving the triad some “sparkle.” (That is, the main beating in the sweet triads is between the third partial of the root tone, with the fundamental of the fifth.) The third being in tune here seems to outweigh the fifth being in tune in the Pythagorean triad, but remember that the third in that case was *much* further out of tune than the fifth is here. The wolf-crossing B, F♯, C♯, and G♯ triads sound pretty rough. This puts the out-of-tune triads in the “raised” keys with many sharps or flats.

(by fifths) (chromatic)

4.5.4.1 1/3-Comma Meantone Temperament

In third-comma meantone, the idea is similar, but remember there is even *more* compression of the fifths to favor the *minor* thirds, which doesn’t help things at all for the *major* triads. All the triads are more out of tune, the “bad” ones being far worse because the wolf fifth is much sharper.

(by fifths) (chromatic)

4.5.4.2 1/5-Comma Meantone Temperament

In fifth-comma meantone, the major thirds and fifths are equally out of tune (Problem 4.4), the thirds being sharp and fifths being flat. The wolf fifth is also a little bit less out-of-tune than in quarter-comma. Even though there are two notes in each triad that are detuned from just, they are So you might expect, this should sound even better than quarter-comma meantone, but remember that the compromise on the third is much bigger than the compromise on the fifth, so it is really better to get rid of the detuned third.

(by fifths) (chromatic)

4.5.4.3 1/6-Comma Meantone Temperament

In sixth-comma meantone, the tempering is gentler than in the other meantones above in order the make the wolfy keys more usable. It is pretty much what you expect: slightly more out-of-tune fifths, except for the wolfy triads, which don’t sound great, but are not quite as bad as in the other cases.

(by fifths) (chromatic)

4.5.4.4 All Together Now

Since there are so many meantones here, let's play the C triad for the different meantones *together*, so you can get a better side-by-side comparison. Here is the C-major chord, for third, fourth, fifth, and then sixth meantone.

Which sounds the best to you? (There is no correct answer here, this is subjective.) We can do the same things for the G♯ "wolf" triad for the same progression of meantones, though here this should become increasingly "nicer."

4.6 Exercises

Problem 4.1

In (quarter-comma) meantone tuning, the comma between enharmonic notes (like G♯/A♭) is called the **enharmonic comma**. Specifically, this is the ratio of the sharp to the flat of the enharmonic pair. Compute the enharmonic comma and show that it is the same as the lesser diesis (see Problems 2.2 and 3.6).

Problem 4.2

The first systematically described temperament was by Zarlino, and amounts to 2/7-comma meantone (Zarlino later described quarter-comma and third-comma meantones).[13]

(a) What is the ratio corresponding to the (regular) perfect fifth in "Zarlino I" (2/7-meantone)? Give the detuning from the just fifth in cents.

(b) What is the wolf fifth? Give the detuning from the just fifth in cents.

(c) What is the (regular) major third? Give the detuning from the just major third in cents.

Problem 4.3(!)

The lowest string on a double bass is typically tuned to E_1 when played as an open string. Assuming the frequency f and length L of a stopped string are related by $f \propto 1/L$, and an open-string length of 110 cm, what is the distance on the fingerboard between the stop positions to play $E\sharp_1$ and F_1, assuming quarter-comma meantone tuning?

Problem 4.4(!)

Recall that Pythagorean temperament achieves justly tuned fifths at the expense of mistuned (sharp) thirds, and (quarter-comma) meantone temperament shifts things to get justly tuned thirds, at the expense of mistuned (flat) fifths.

Deduce the meantone temperament that represents an *equal* compromise between mistuned thirds and fifths. That is the thirds and fifths are the same "distance" from their justly tuned counterparts, in the sense of the same number of cents (provided we stay far enough away from the wolf interval). Compute the fifth and third intervals in this temperament, as well as the "comma," or difference of these intervals from the just ratios (express as both ratios and in cents).

Problem 4.5(!)

(a) Justify the notion that equal temperament (12-EDO) is a particular case of a meantone temperament. Which meantone is it? (That is, if 12-EDO is $1/n$-meantone, what is n?)

(b) Based on (a), justify the notion that 1/11-meantone is a close approximation to 12-EDO. What is the difference in fifths (in cents) in the two tunings? What is the difference (in cents) between C and B♯ in each tuning?

Problem 4.6(!)

Which meantone has a "wolf" fifth that is *just*?

[13] Zarlino, *Istitutioni*, Part II, p. 125 (1558); as cited in Rudolf Rasch, *op. cit.*

Chapter 5

Unequal Closed Temperaments

We have seen that the various meantone temperaments improve the quality of thirds, but at the expense of introducing a wolf fifth. The main problem is that certain meantone keys (F♯, C♯, G♯) become off-limits, restricting the possibilities for transposing or modulating keys in modern music. One approach to improving this situation is to "spread out" the extra wolf comma in the wolf fifth around the circle of fifths. Equal temperament accomplishes exactly this, by spreading out the comma equally over all the fifths. Again, one disadvantage here is that the major thirds are still not especially pure. Many other temperaments with *unequal* spacings of the fifths have also been developed and used, to eliminate the wolf fifth and make various compromises to other intervals. Such temperaments in which all keys are usable are called **closed temperaments**, **well temperaments**, or **circulating temperaments**. The unequal spacings allow more purity of important intervals in some keys, while limiting dissonance in other keys to an acceptable level. Equal temperament is of course completely uniform in every key, which could be an advantage or a disadvantage—for example, an unequal temperament has different levels of "sweetness" or "spice" in different keys, which can be employed for musical effect by a skilled composer. A famous example (and the subject of much debate) is Bach's *Well-Tempered Clavier*, which likely refers to an unequal well temperament, not equal temperament.

Famous examples were temperaments by Werckmeister, Kirnberger, Neidhardt, and Vallotti. We will survey a few of the more historically important and interesting closed temperaments here.[1,2]

5.1 Notation

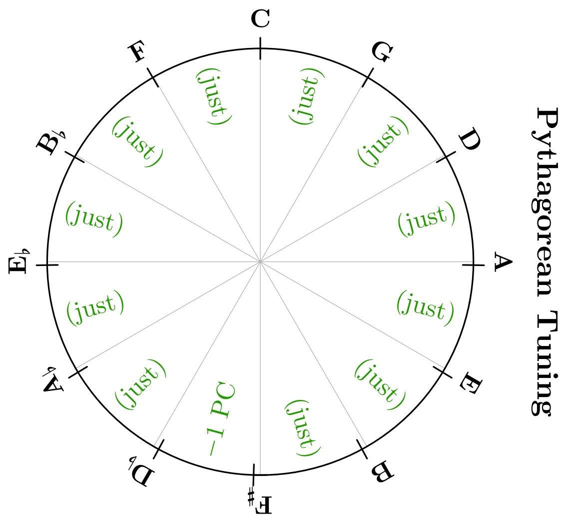
The temperaments we will examine here are derivatives of Pythagorean tuning, in the sense of being tweaks of the basic Pythagorean stack of fifths. The differences among the temperaments involve how (or if) each fifth is modified from the just interval. Thus, a circle of fifths (with the modifications indicated) suffices to define each temperament.

5.1.1 Pythagorean Tuning

For example, we can represent Pythagorean tuning as in the circle below. As in the previous circle-of-fifths diagrams, the "pie-slices" are into equal parts (i.e., 12-EDO divisions), with the positions of the notes reflecting the deviations of the intervals from the equal divisions (the deviations are exaggerated to make them more visible).

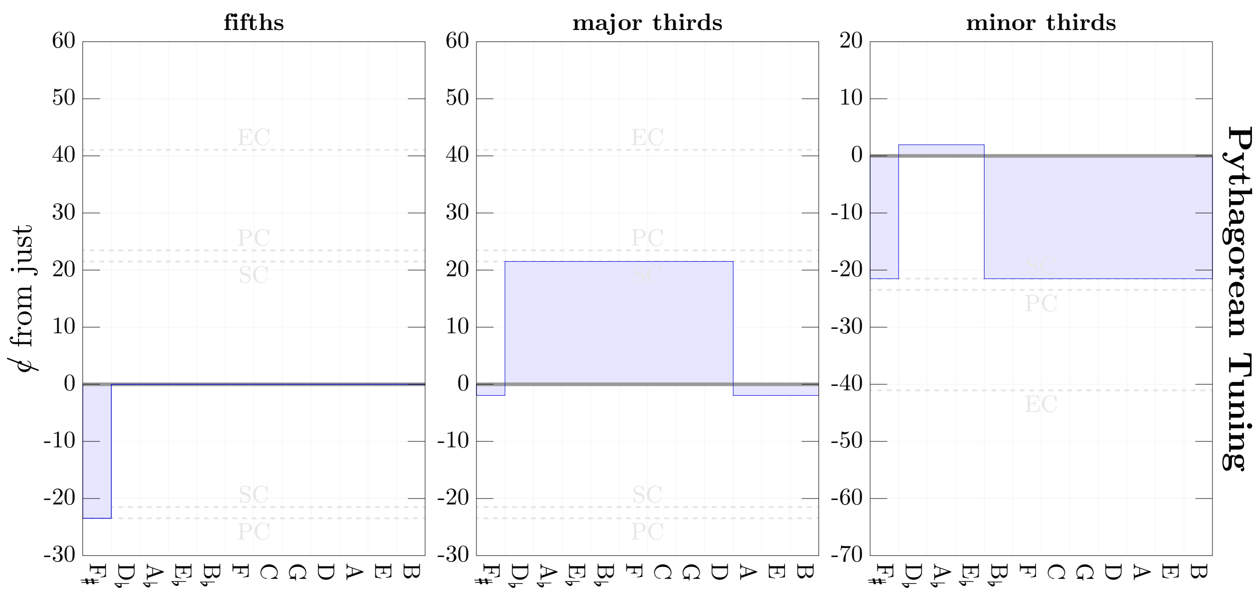
[1] For more comprehensive catalogs of temperaments, good references (albeit not in English) are Pierre-Yves Asselin, *Musique et Tempérament* (Éditions Costallat, 1985); and Jos de Bie, "Historische orgelstemmingen: theorie en indeling," in Hans Fidom, ed., *Hoe Mooi Huilt de Wolf? Orgelstemmingen in de tijd van Albertus Anthoni Hinsz* (Stichting Groningen Orgelland, 2005), p. 11.

[2] Another good source of information on temperaments, especially on how to tune them, is the technical library at the site of Carey Beebe Harpsichords: `http://www.hpschd.nu/tech/index.html`.



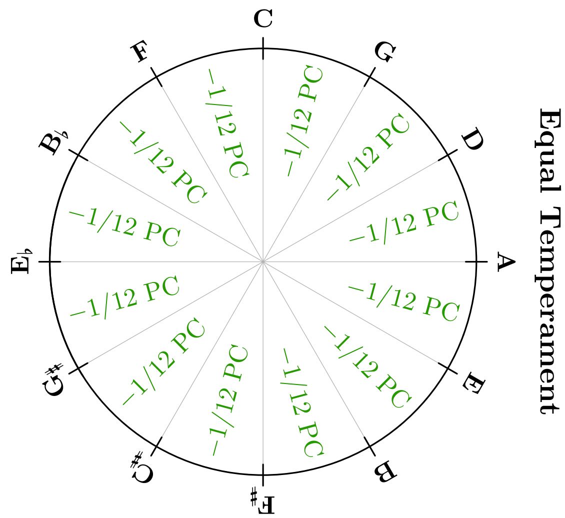
Here, all fifths are just, except for the wolf fifth. Remember that the main problem in Pythagorean tuning comes from the Pythagorean comma [Eq. (2.5)], which is the mismatch between 12 just fifths and 7 octaves. This comma must end up somewhere, and in Pythagorean tuning it all ends up in the wolf fifth. In the diagram, we indicate this by a $-1\,\mathrm{PC}$, to indicate that the interval is *narrow* compared to a pure fifth by one Pythagorean comma. Remember the PC is $3^{12}/2^{19}$, and so a pure fifth less the PC is $(3/2)/(3^{12}/2^{19}) = 2^{18}/3^{11} = R_{\mathrm{wolf}}$, which matches what we calculated before in Eq. (2.2).

One way to characterize a temperament, is how far the consonant intervals deviate from their just ratios. You can deduce this by looking at the circle of fifths, but below are graphs that show all the fifths, major thirds, and minor thirds, and their deviations in cents from the just ratios (3/2, 5/4, and 6/5, respectively). The Pythagorean comma, syntonic comma, and enharmonic comma (diesis) are also marked for reference. This makes it easy to compare the various temperaments, as well as to see quickly which intervals should sound good or bad. Here, we can clearly see the wolf fifth, the many sharp major thirds, and the many flat minor thirds.

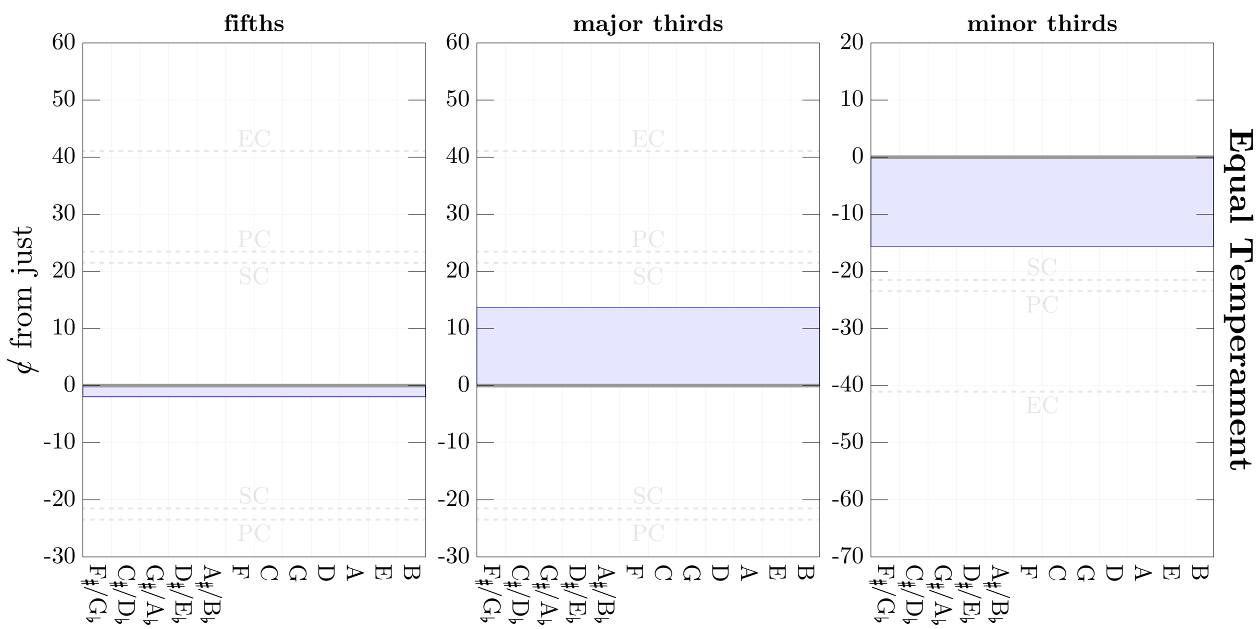


5.1.2 Equal Temperament

As a second example, recall that equal temperament fixes the wolf fifth by taking the "extra stuff" ($-1\,\mathrm{PC}$), and distributes it evenly through all the fifths.

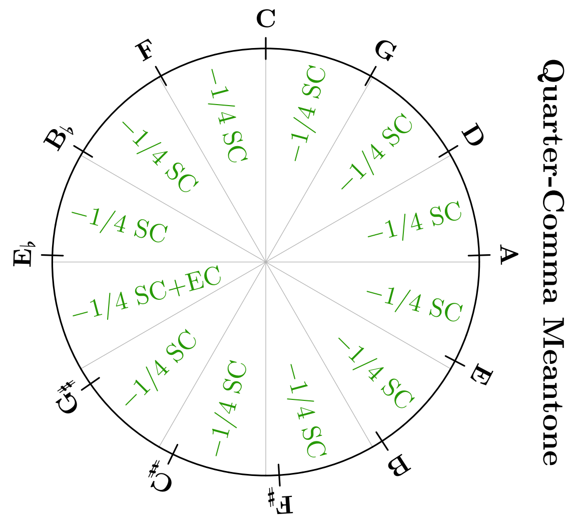


That is, each interval is tempered by $-1/12\,\mathrm{PC}$. The corresponding set of graphs makes it clear that the keys are all equivalent, and that the thirds are worse off than the fifths.



5.1.3 Meantone Temperament

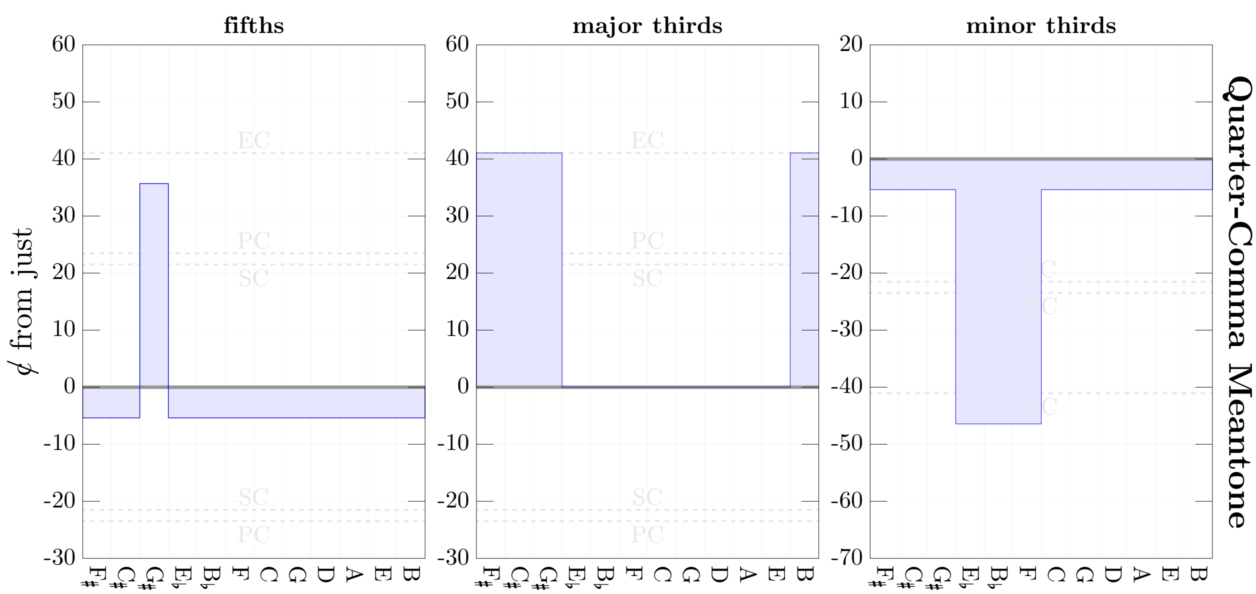
As a third example, consider quarter-comma meantone temperament (Section 4.2), as in the circle of fifths below.



All but one of the fifths are narrowed from the just fifth by 1/4 of the syntonic comma (SC), as in Eq. (4.2), hence the notation −1/4 SC. The wolf fifth between G♯ and E♭ is a pure fifth less 1/4 a syntonic comma, but then *widened* by the **enharmonic comma** (Problem 4.1). We will leave it as an exercise to show that this combination works out precisely (Problem 5.1), but roughly speaking, it works out as follows. The temperings, not including the EC, are a total of 12 quarter-commas, or 3 times the SC. The SC is the difference (ratio) between a Pythagorean and just major third, and so 3 SC is the ratio of 12 just fifths to 3 just thirds. The enharmonic comma is the same as the diesis (Problem 2.2), which is the ratio of the octave to three major thirds. So adding the EC to the −3 SC gives −1 PC, which is the correction we need to distribute around the circle.

A more literal "spelling" of the wolf interval is $+(11/4)\,\mathrm{SC} - 1\,\mathrm{PC}$, because we need to end up with a net comma of −1 PC around the circle, so we can just "undo" the SC tempering from the 11 other intervals, and then introduce the PC. This is sometimes listed as a net +(7/4) SC, but this is only approximately true (this comes from saying SC ≈ PC, which is only correct to about 9%).

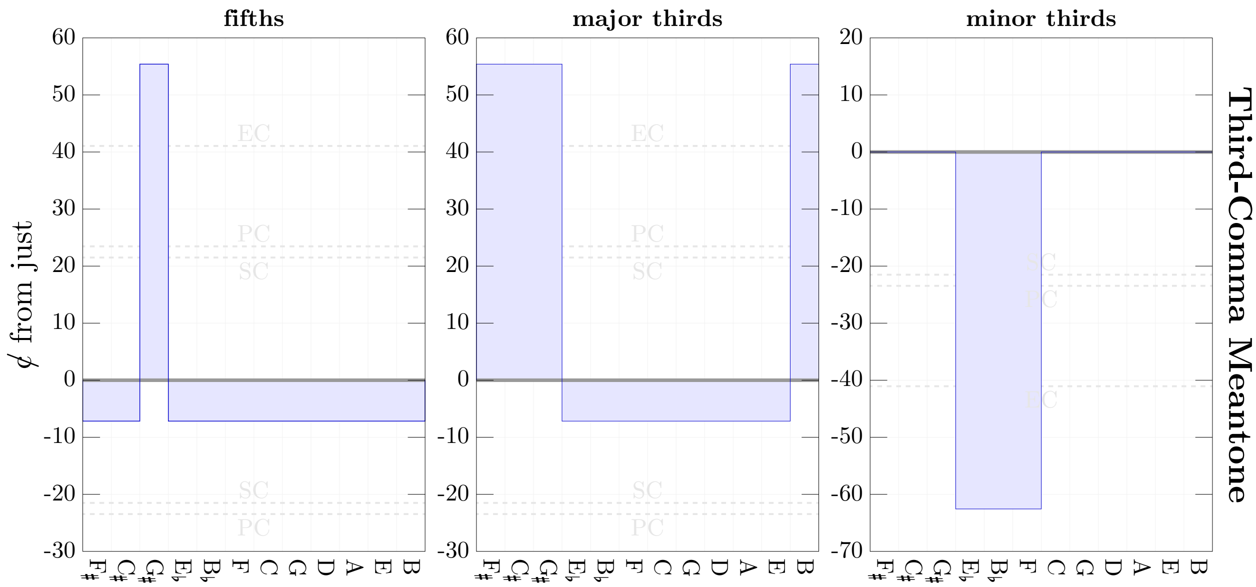
From the graphs, we can see that the wolf fifth is really out of tune, as are a few of the major and minor thirds, but the majority of the major thirds are pure.



5.1.3.1 Other Meantones

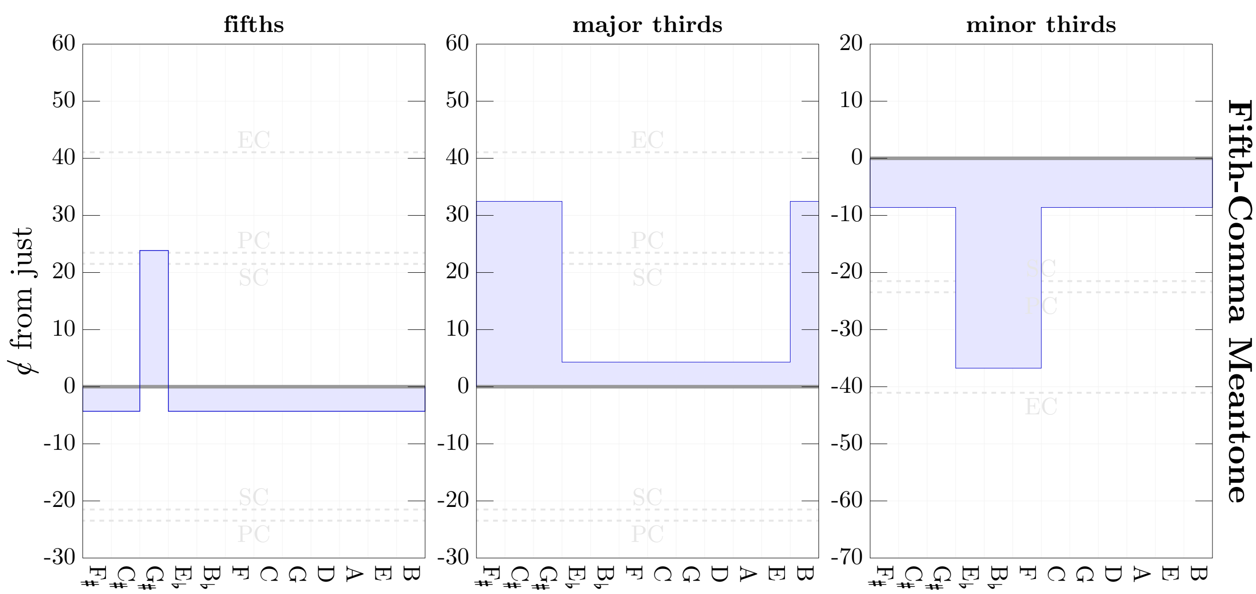
Of course, in other meantones, the basic pattern is the same, but exactly how much the intervals differ from their pure values differs for each temperament.

Third-Comma Meantone:



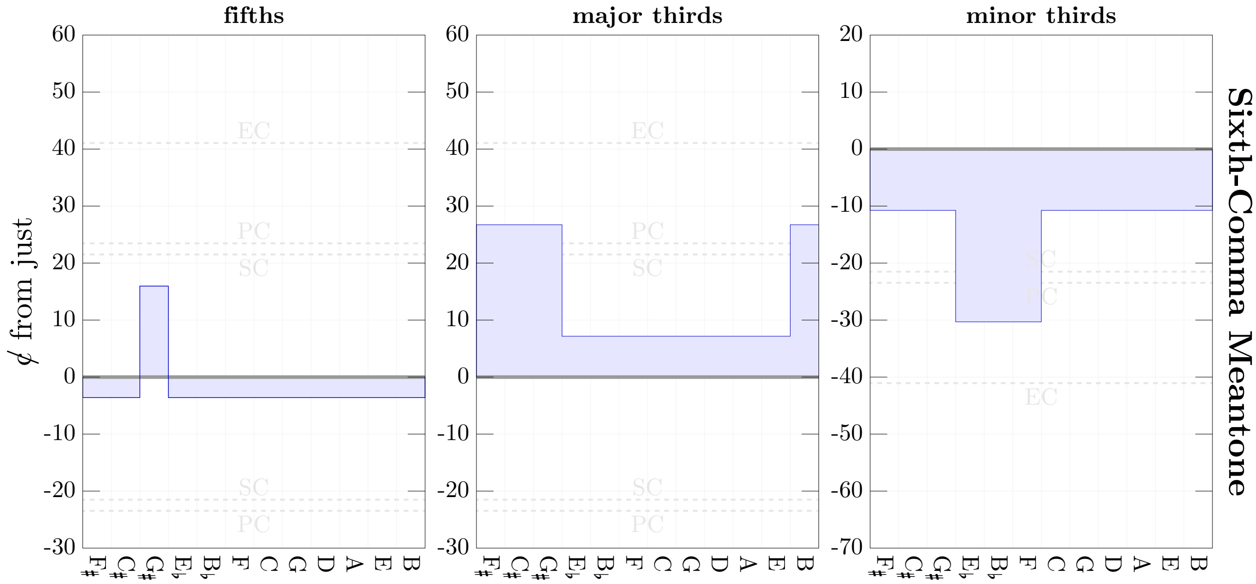
Again, for third-comma meantone, most of the minor thirds are just, but the deviations of the “bad” intervals are really quite bad. The wolf fifth, for example, is closer to the augmented fifth than to the perfect fifth. The bad thirds are similarly off by more than a quarter tone.

Fifth-Comma Meantone:



Here, the bad intervals are relatively more mild than in quarter-comma meantone.

Sixth-Comma Meantone:



The bad intervals are even more mild than in quarter- and fifth-comma meantone, the tradeoff, of course, being less pure major thirds.

5.2 Werckmeister III

Andreas Werckmeister (1645–1706) originated the idea of **concentric tunings**—closed temperaments that better optimize "central" keys (keys with no or few sharps/flats).[3] He published several temperaments in 1691,[4] the most famous being his "Num. 3," commonly called **Werckmeister III**;[5] he also described this temperament earlier in 1681.[6]

Werckmeister intended this temperament to be a relatively simple retuning of quarter-comma-meantone pipe organs to a circulating temperament. He specified that four fifths should be compressed by 1/4 of a comma, though he did not specify exactly *which* comma.[7] But since the quarter-commas must account for the Pythagorean comma, these fifths must each be tempered by one quarter of the Pythagorean comma. The Werckmeister III scheme is shown in the circle of fifths below.

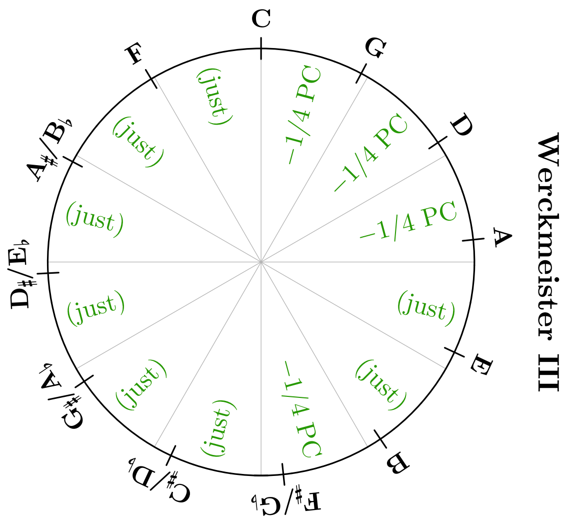
[3]Rudolf Rasch, "Tuning and temperament," in *The Cambridge History of Western Music Theory*, Thomas Christensen, Ed. (Cambridge, 2002), p. 193 (ISBN: 0521686989) (doi: 10.1017/CHOL9780521623711).

[4]Andreas Werckmeister, *Musicalische Temperatur* (1691), available online at `http://imslp.org/wiki/Musicalische_Temperatur_(Werckmeister,_Andreas)`.

[5]Werckmeister, *op. cit.*; see the top half of p. 78.

[6]Andreas Werckmeister, *Orgel-Probe* (1681); the later edition of this work, *Erweierte und verbesserte Orgel-Probe* (1698) is available online at `http://imslp.org/wiki/Orgel-Probe_(Werckmeister,_Andreas)`.

[7]Paul Poletti, "Temperament and intonation in ensemble music of the late eighteenth century: performance problems then and now," in *Music of the past — instruments and imagination*, Michael Latcham, Ed. (Peter Lang, 2006), p. 109 (ISBN: 3039109936).

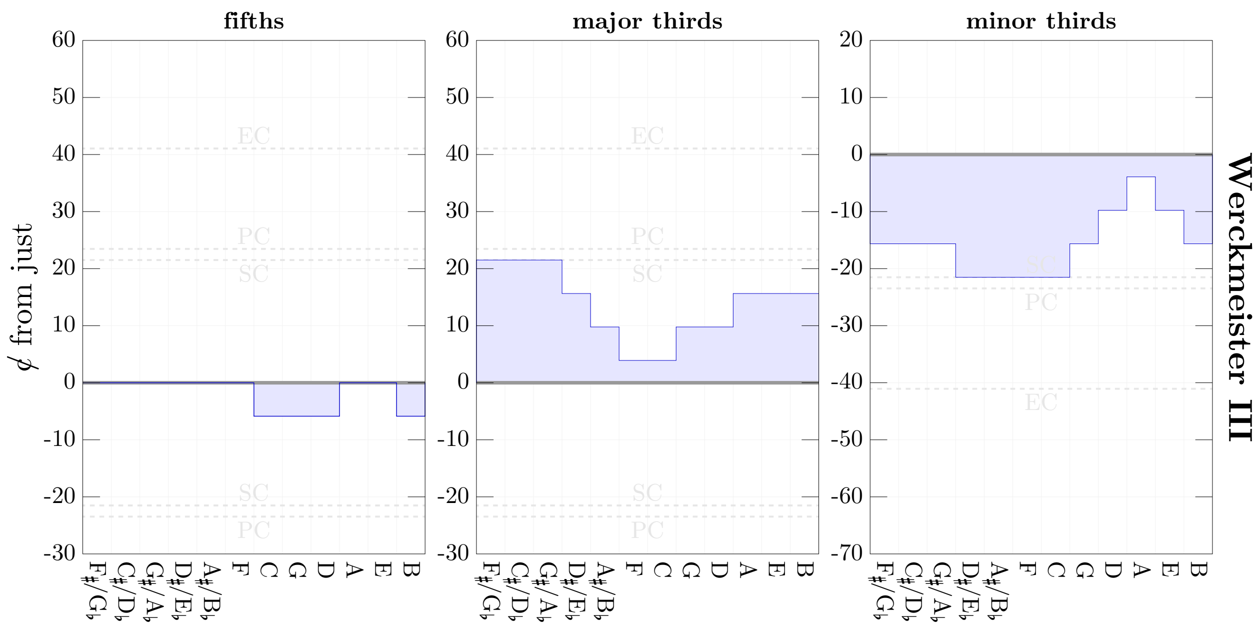


Note that if we ignore the (fairly small) difference between the Pythagorean and syntonic commas, this temperament has four fifths that are essentially meantone fifths, but with the rest of the intervals "opened up" to just—the economy here in retuning many pipes in an organ from meantone follows from leaving the four meantone fifths untouched.

Notice that unlike meantone, there are no pure major thirds: a pure major third requires four *consecutive* compressed fifths (and by 1/4 SC, not 1/4 PC). However, the "main" C–E third is not too far off; the ratio is the Pythagorean major third (81/64) compressed by 3/4 of the Pythagorean comma, or

$$R_{\text{C–E}} = \frac{81}{64}\left(\frac{2^{19}}{3^{12}}\right)^{3/4} = \left(\frac{5}{4}\right)\left(\frac{1024\sqrt[4]{2}}{1215}\right) = \left(\frac{5}{4}\right)(+3.9\,¢)\,, \tag{5.1}$$

or about 4 ¢ sharp of pure. In three of the less-central keys (C♯, G♯, E♭), the major thirds open up to Pythagorean thirds. The fifths are either just or close to the meantone value (1/4 PC is about 5.9 ¢).



Werckmeister III is still used in modern, baroque-style pipe organs. For example, organ builder John Brombaugh was a pioneer in reintroducing unequal temperaments back into pipe organs after an era of equal temperament.[8] He favored Werckmeister III in his earlier organs; a specific exmaple is the "Opus 11a" organ

[8]Marga Jeanne Morris Kienzle, *The Life and Work of John Brombaugh, Organ Builder*, D.M.A. thesis (1984).

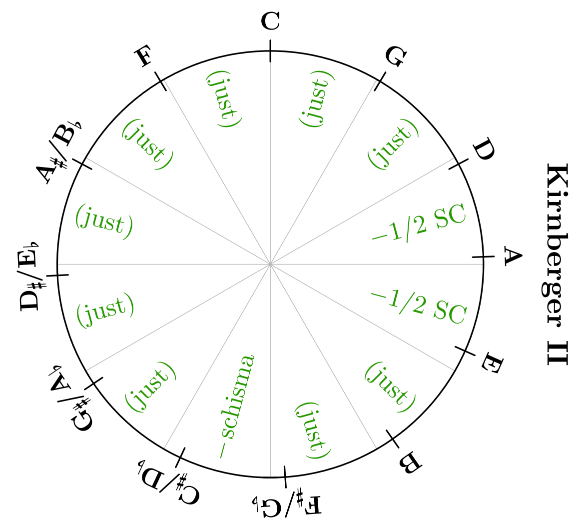
(1973) at the Oberlin Conservatory in Oberlin, Ohio.[9]

To continue the listening examples from Section 4.5, here are all the major triads in Werckmeister III, by fifths and chromatically.

(by fifths) (chromatic)

5.3 Kirnberger II and III

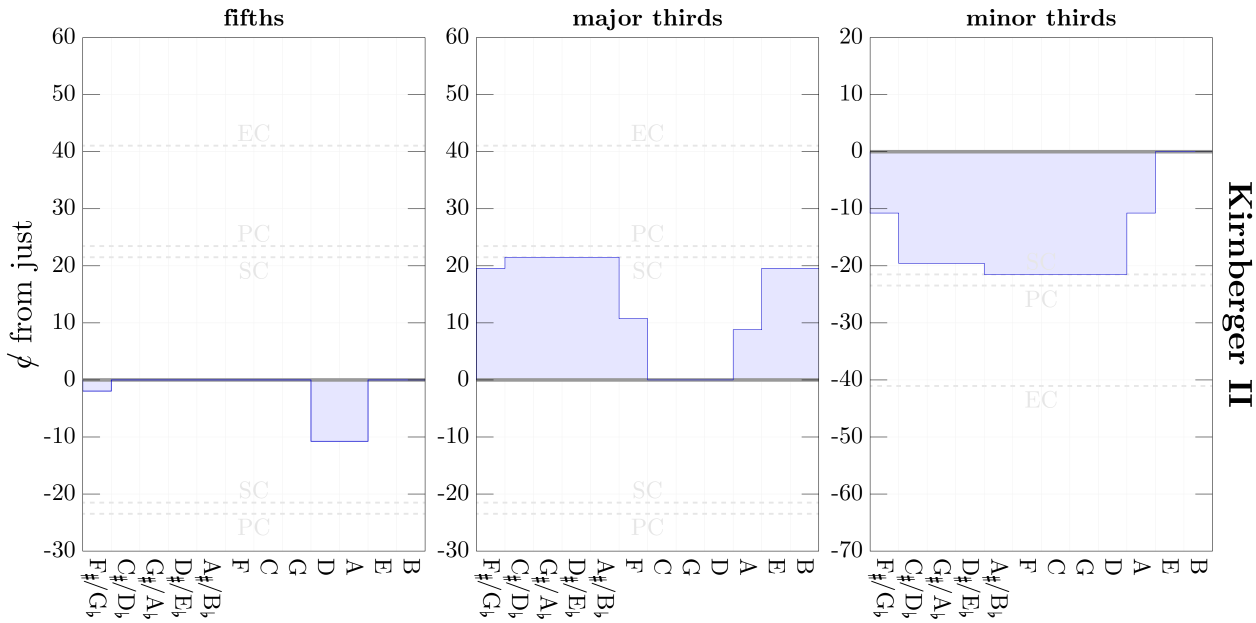
Johann Philipp Kirnberger (1721Ð1783) was a composer, music theorist, and (briefly) a student of Bach. In 1771 he published what is now called **Kirnberger II**.[10] The scheme is shown in the circle of fifths below.



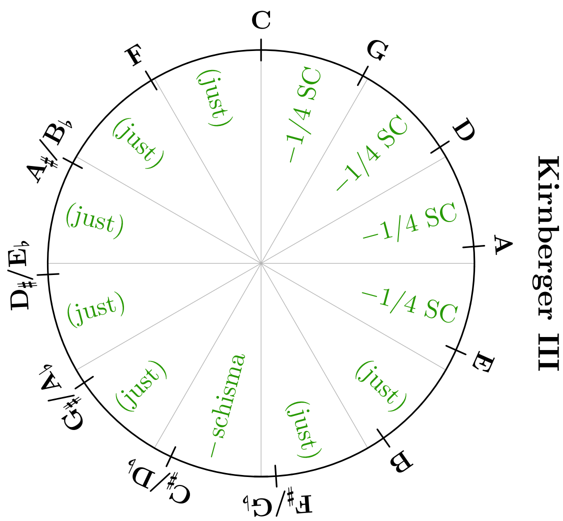
The idea is to take the syntonic comma, split it in half, and temper two intervals by the half commas. But remember that they circle of fifths has to bury a *Pythagorean comma*, not a syntonic comma. The difference (ratio) between the Pythagorean and syntonic commas is the **schisma** (Problem 2.3), which is about 2 ¢ (Problem 3.7). Since the two fifths are *compressed* by a total of the syntonic comma, one other interval (tucked away far from C) must similarly be compressed by the schisma to add up to a total compression of the Pythagorean comma.

[9] Homer Ashton Ferguson III, *John Brombaugh: The Development of Americas Master Organ Builder*, D.M.A. dissertation (2008), p. 104.

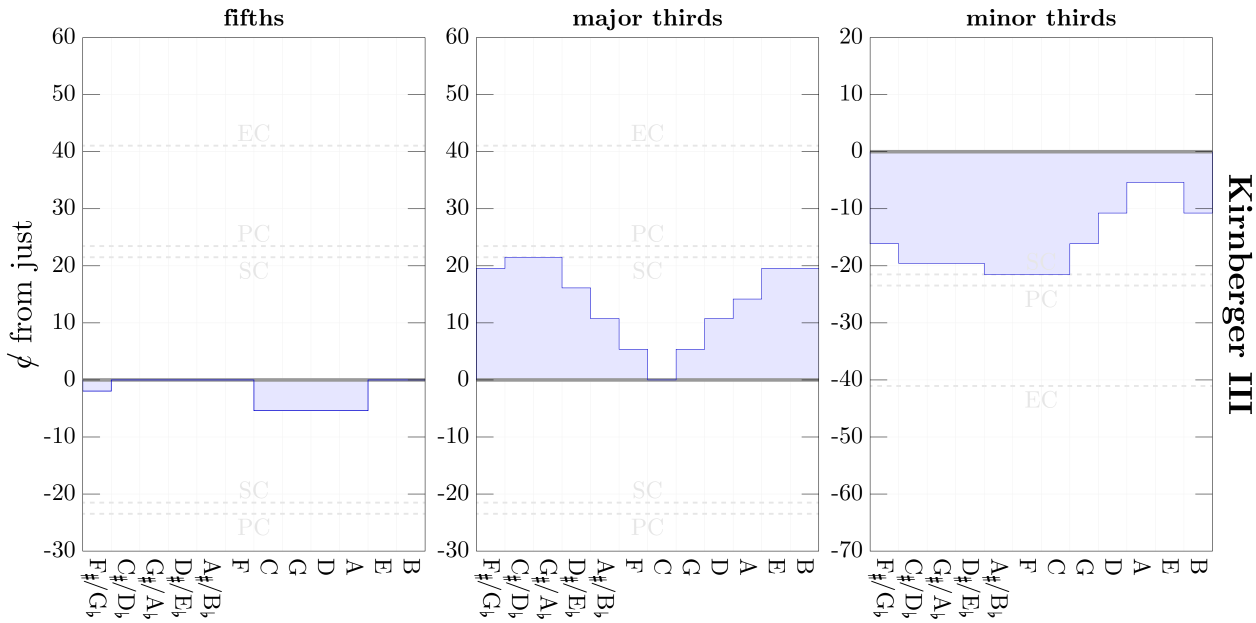
[10] Johann Philipp Kirnberger, *Die Kunst des reinen Satzes in der Musik* (1771), part I, p. 13; available online at `http://imslp.org/wiki/Die_Kunst_des_reinen_Satzes_in_der_Musik_(Kirnberger,_Johann_Philipp)`



The advantage of Kirnberger II is three pure major thirds (C–E, G–B, D–F♯). However, the fifths D–A and A–E are far from just (half of a syntonic comma is almost 11 ¢, so these fifths are significantly flat). This problem with the fifths motivated him to publish a modified temperament in 1779, now called **Kirnberger III**. The scheme for this temperament is shown below.



The fifth here have a less severe tempering: the syntonic comma is split over *four* fifths, and the schisma is still in place to make the octave come out right. There is only one pure major third (C–E), while the others more gradually expand up to Pythagorean thirds. The fifths are either just or the same as in meantone. In central keys, this temperament is more similar to meantone, while in “raised” keys (keys with more sharps/flats) this temperament is more Pythagorean (but without the wolf).



Harpsichord builder Carey Beebe calls Kirnberger III "one of the easiest—and most practical—temperaments to set"[11] and "one of the best temperaments for first-time tuners to gain confidence with."[12] A number of pipe organs are tuned to Kirnberger temperaments; for example, organ builder John Brombaugh switched from Werckmeister III to a modified version of Kirnberger III,[13] before switching again to Kellner–Bach later in his career.

The major triads in Kirnberger II, are here by fifths and chromatically,

(by fifths) (chromatic)

as are the triads for Kirnberger III:

(by fifths) (chromatic)

5.4 Neidhardt II

Johann Georg Neidhardt (1680–1739) was interesting in recommending different temperaments for different settings. He believed that in a large city, with larger organs, playing more diverse and modern literature, a more equal temperament is appropriate, for example, than in a village.[14] For example, in 1724, Neidhardt published four temperaments.[15,16] He described them, from least to most equal (the last one being equal temperament), as follows:

[11] `http://www.hpschd.nu/tech/tmp/kirnberger.html`

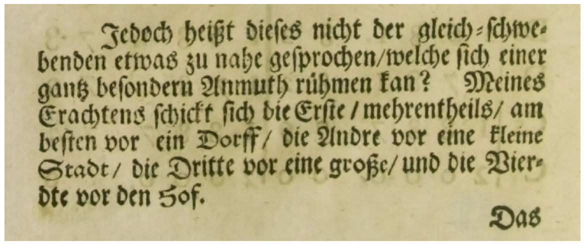
[12] `http://www.hpschd.nu/tech/tmp/kirnberger-2.html`

[13] Homer Ashton Ferguson III, *op. cit.*, p. 64.

[14] Jos de Bie, *op. cit.*, p. 18.

[15] Johann Georg Neidhardt, *Sectio canonis harmonici* (1724), available online at `http://imslp.org/wiki/Sectio_canonis_harmonici_(Neidhardt,_Johann_Georg)`.

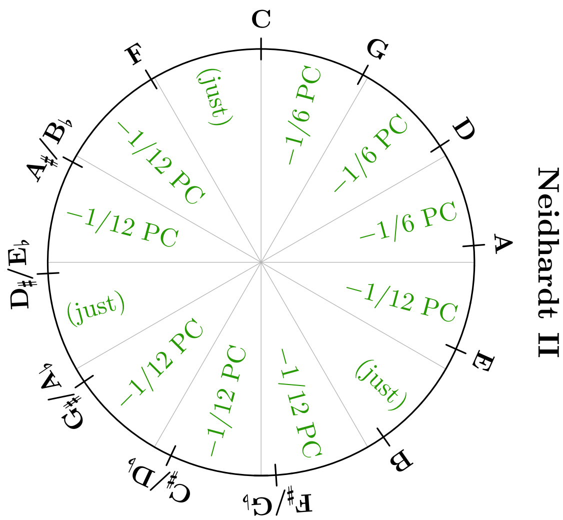
[16] For a list of the temperaments, see Rudolf Rasch, *op. cit.*, pp. 218-9.



Jedoch heißt dieses nicht der gleich-schwe-
benden etwas zu nahe gesprochen/welche sich einer
gantz besondern Anmuth rühmen kan? Meines
Erachtens schickt sich die Erste/mehrentheils/am
besten vor ein Dorff/ die Andre vor eine kleine
Stadt/ die Dritte vor eine große/ und die Vier-
dte vor den Hof.
Das

This passage translates to something like: "However, does this not mean something too close to equal temperament, which can pride itself on quite a particular grace? In my opinion, the first is suited, for the most part, best for a village, the second for a small town, the third for a big city, and the fourth for the court."

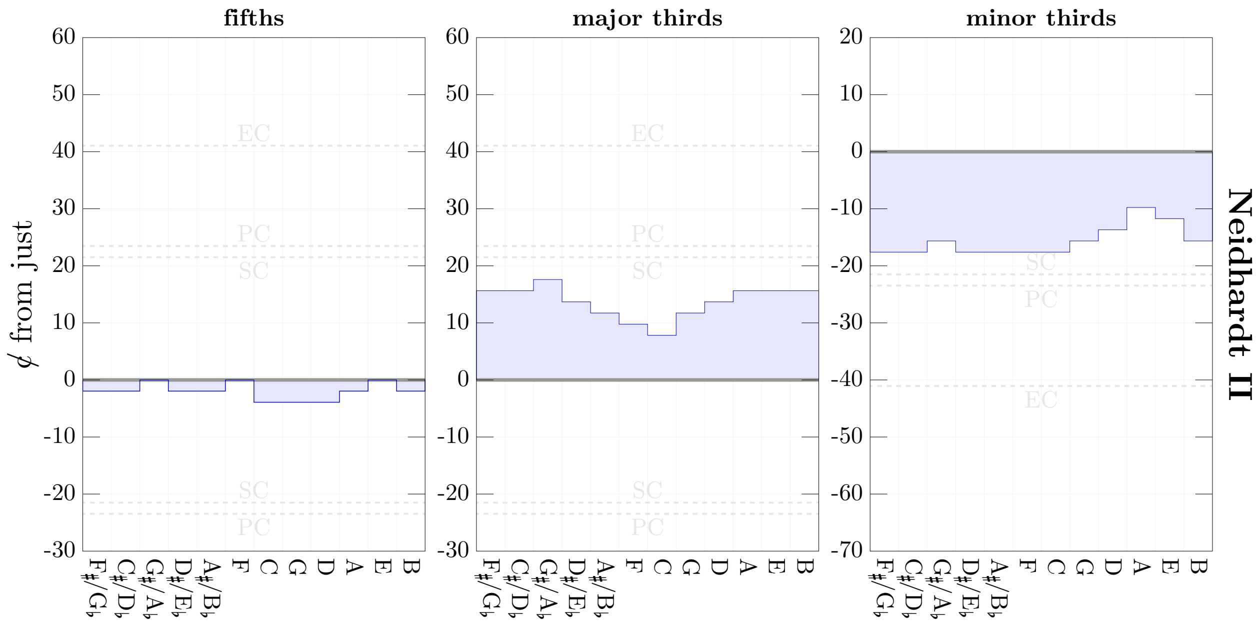
We will look at one of these temperaments, now called **Neidhardt II**, or the 1724 *Kleine Stadt* (small town). This temperament was later repurposed in 1732[17] for a *Große Stadt*, or big city; in fact all his temperaments shifted towards the uneven end in this publication.[18] The circle-of-fifths scheme is shown below.



Here the Pythagorean comma is divided into three "small bits" and 6 more "half small bits." The C-major triad is close to sixth-meantone, except that the third is tempered slightly less. Remember that "−1/12 PC" is the same as an equal-tempered fifth, so the raised keys are similar to equal temperament, while the central keys are tempered slightly more to favor the major thirds. There are a few just fifths, but no just thirds. Overall, the temperament here is closer to equal than Werckmeister III or the Kirnbergers.

[17] Johann Georg Neidhardt, *Gäntzlich erschöpfende, mathematische Abtheilungen des Monochordi* (1732). The 21 temperaments from this publication are listed here: `http://harpsichords.pbworks.com/f/Neidhardt_1732_Charts.pdf`.

[18] Jos de Bie, *op. cit.*, p. 18



The major triads in Neidhardt II, are here by fifths and chromatically:

(by fifths) (chromatic)

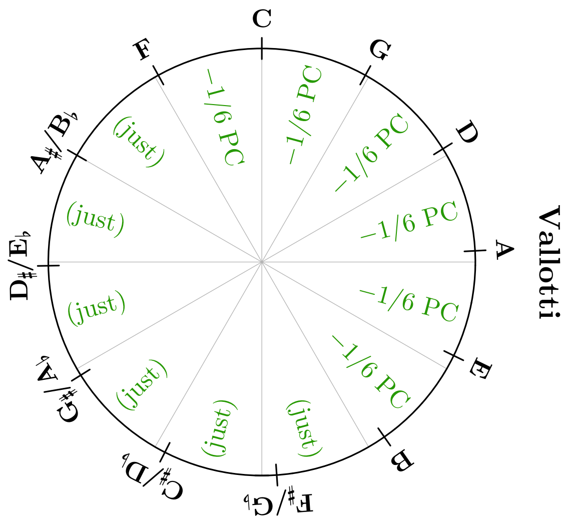
5.5 Vallotti

Francesco Antonio Vallotti (1697–1780) was a composer and music theorist. He devised the temperament commonly called **Vallotti temperament** by 1779.[19] This temperament was first described in print by another music theorist, Giuseppe Tartini (1692-1770) in 1754,[20] and thus this temperament is also called **Tartini–Vallotti temperament**.[21] The circle of fifths for Vallotti is shown below.

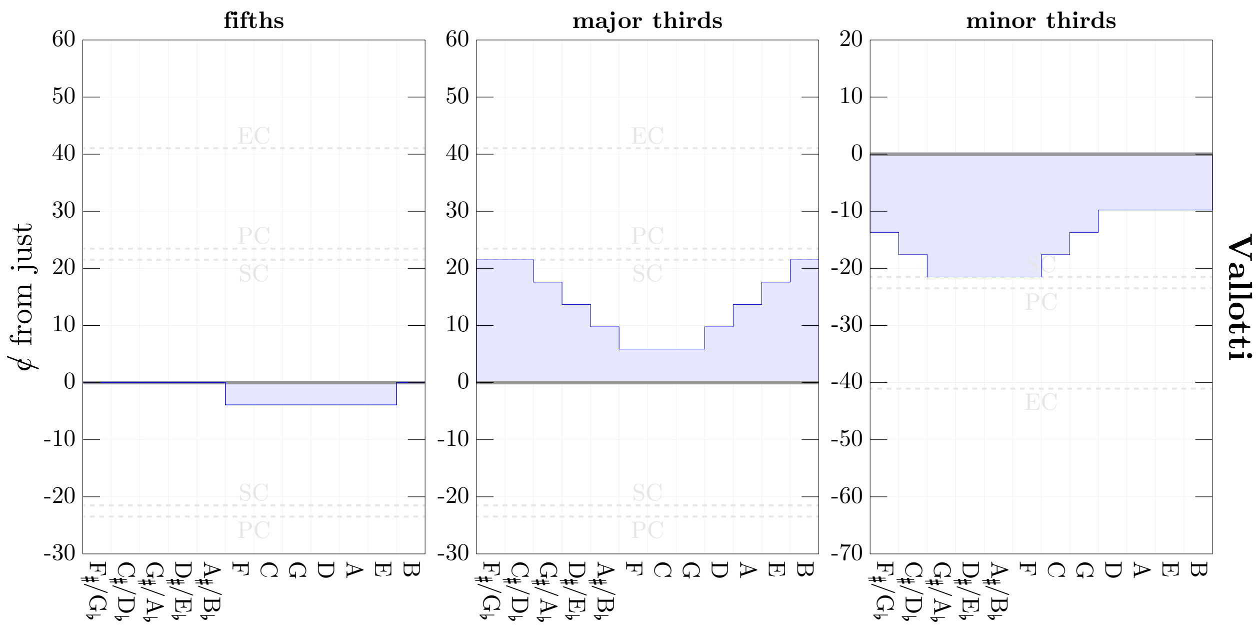
[19]Francesco Antonio Vallotti, *Della scienza teorica e pratica della moderna musica* (1779), but only volume 1 (of 4) was published here. The temperament is in volume 2, published as Francesco Antonio Vallotti, *Trattato della moderna musica* (Il Messagero di S. Antonio, 1950). See Chapter IV, especially p. 197.

[20]Giuseppe Tartini, *Trattato di musica secondo la vera scienza dell' armonia* (1754), available online at `https://archive.org/details/trattatodimusica00tart`. See p. 100 and the reference to Vallotti there.

[21]Pierre-Yves Asselin, *op. cit.*; note that what Asselin calls "Tartini–Vallotti" is the "Young II" temperament in the next section.



Here, the Pythagorean comma is spread into six parts, for a fairly equal temperament, compared to the other unequals. There are three major thirds (F–A, C–E, G–B) that are tempered by $-2/3$ PC, three more that are Pythagorean, and the rest gradually interpolating between the extremes.



Because of the evenness of this temperament, Vallotti is regarded as versatile. Harpsichord builder Carey Beebe states[22] "Vallotti is one of the most often requested temperaments for ensemble use in late baroque and classical repertoire." However, elsewhere he also points out[23] "Some musicians despise Vallotti's popularity as a generic all-purpose temperament." Author Ross Duffin appears to be one of these musicians, with an article titled 'Why I hate Vallotti (or is it Young?),"[24] which argues for sixth-meantone as an improvement.

The major triads in Valotti, are here by fifths and chromatically:

(by fifths) (chromatic)

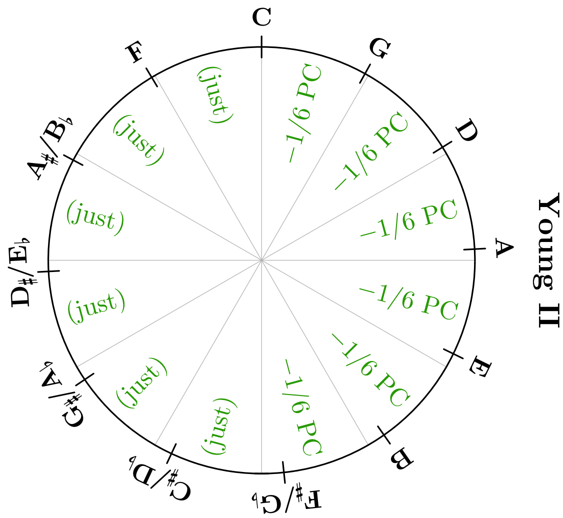
[22] `http://www.hpschd.nu/tech/tmp/vallotti.html`

[23] `http://www.hpschd.nu/tech/tmp/neidhardt-gs.html`

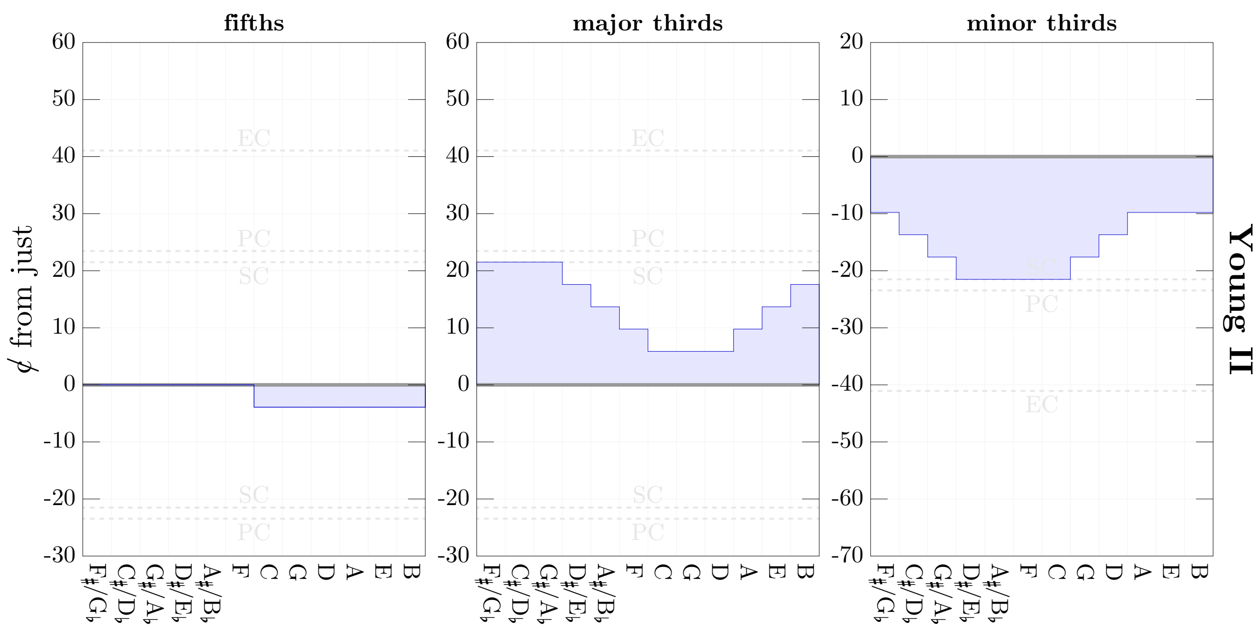
[24] Ross W. Duffin, "Why I hate Vallotti (or is it Young?)" `http://musicserver.case.edu/~rwd/Vallotti/T1/page1.html`.

5.5.1 Young II

Thomas Young (1773-1829) was a scientist who was most famous for demonstrating the wave nature of light (in the Young double-slit experiment) and for his work in translating the Rosetta stone. He published a musical temperament, now called **Young II**,[25] in 1800. The scheme is shown below.



Essentially, this is a transposition of Vallotti: the tempered fifths start on C instead of on F. While it is possible to argue that Vallotti and Young are very different,[26] the two temperaments are very similar.



The major triads in Young II, are here by fifths and chromatically:

(by fifths) (chromatic)

[25] Thomas Young, "Outlines of Experiments and Inquiries Respecting Sound and Light. By Thomas Young, M. D. F. R. S. In a Letter to Edward Whitaker Gray, M. D. Sec. R. S.," *Philosophical Transactions of the Royal Society of London for the year 1800* **90**, 106 (1800) (doi: 10.1098/rstl.1800.0008). See the section "Of the temperament of musical intervals," p. 143.
[26] Ross W. Duffin, *op. cit.*

5.6 Bach: *The Well-Tempered Clavier*

An interesting and famous example of a closed temperament is *The Well-Tempered Clavier* (*Das Wohltemperierte Clavier*, or *Das Wohltemperirte Clavier* in the original spelling) of Johann Sebastian Bach (1685–1750). This was a book of preludes and fugues for every major and minor key (24 total prelude/fugue pairs) in 1722. A similar, second book in 1742 brought the total to 48 prelude/fugue pairs in all keys, in what now stands as a single great work.

The question is, what exactly does *wohltemperierte* mean? Unfortunately, Bach didn't leave an explanation or an explicit set of tuning instructions. Many people have assumed that the 48 were meant to be played in equal temperament, and these pieces have certainly been recorded in equal temperament.[27] For example, Barbour summarized this point of view in 1951:[28] "An equal temperament was needed for the keyboard works of Bach, both for clavier and for organ. It is generally agreed that Bach tuned the clavier equally." However, he also noted that Werckmeister (a contemporary of Bach), used the term *wohltemperierte* specifically for an *unequal*, closed temperament. The German phrase for equal temperament, on the other hand was *die gleichschwebende Temperatur* ("the equal-beating temperament").[29] The prevailing opinion nowadays is that the intended temperament is unequal, although it is not completely clear *which* closed temperament is the "correct" one.

Here we will look at one recent and famous attempt by Bradley Lehman to deduce the temperament from the title page of *The Well-Tempered Clavier*.[30] There are critics of his approach,[31] and due to some of the underlying assumptions, it is (necessarily) impossible for his temperament to be definitive. However, the basic idea is so compelling that it is worth examining. We will only superficially review this work; in addition to the reference to the circle of fifths, the argument for the temperament includes a comparison to historical practice and a subjective analysis of how the *The Well-Tempered Clavier* pieces sound in this temperament. Also, there has been much preceding work on the temperament, as well as criticisms, arguments, and alternate treatments, sufficient to fill a good part of a Ph.D. thesis.[32] Lehman himself gives a good online survey of the various other "Bach" temperaments.[33]

The basic instrument for getting at Bach's intended temperament is the title page of *The Well-Tempered Clavier* shown below,[34] specifically the decoration across the top of the page, above the title.

[27]For example, a well-regarded harpsichord recording in equal temperament on compact-disc is Kenneth Gilbert, *J. S. Bach: Das Wohltemperierte Clavier 1&2* (Polydor International GmbH, 1984).

[28]J. Murray Barbour, *Tuning and Temperament: A Historical Survey*, 2nd ed. (Michigan State College, 1953) (ISBN: 0486434060), p. 195. First edition available online at `https://archive.org/details/tuningtemperamen00barb`.

[29]J. Murray Barbour, *op. cit.*, p. 194.

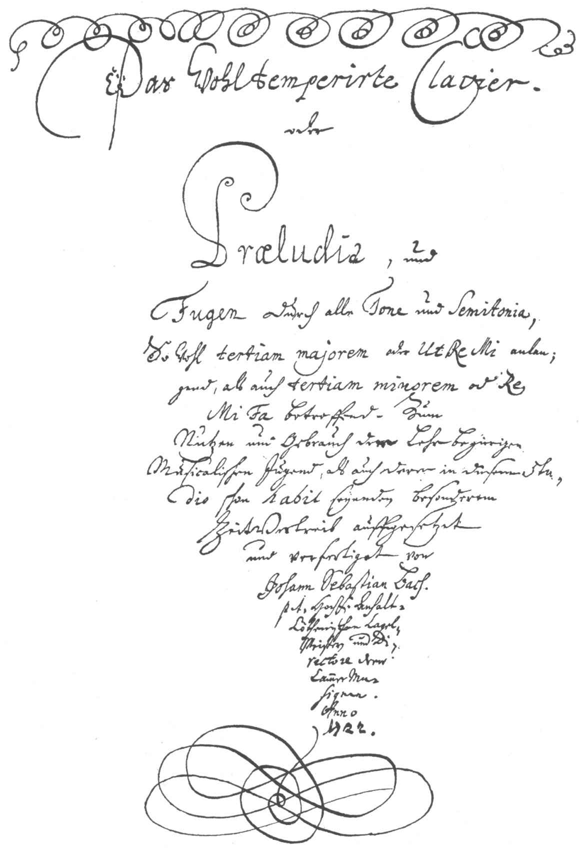
[30]Bradley Lehman, "Bach's extraordinary temperament: our Rosetta Stone—1," *Early Music* **33**, 3 (2005) (doi: 10.1093/em/cah037); Bradley Lehman, "Bach's extraordinary temperament: our Rosetta Stone—2," *Early Music* **33**, 211 (2005) (doi: 10.1093/em/cah067).

[31]See, for example, Mark Lindley and Ibo Ortgies, "Bach-style keyboard tuning," *Early Music* **34**, 613 (2006) (doi: 10.1093/em/cal065); John O'Donnell, "Bach's temperament, Occam's razor, and the Neidhardt factor," *Early Music* **34**, 625 (2006) (doi: 10.1093/em/cal101).

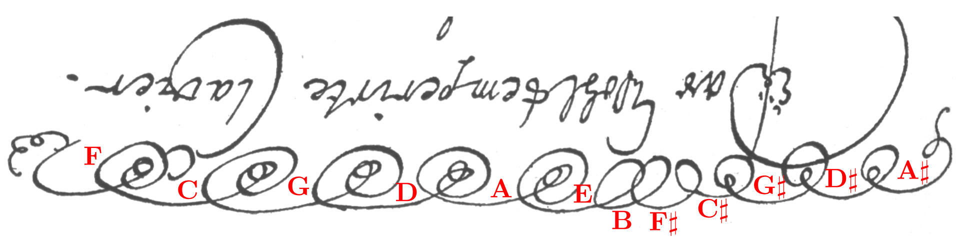
[32]Sergio Martínez Ruiz, *Temperament in Bach's* Well-Tempered Clavier*: A historical survey and a new evaluation according to dissonance theory*, Ph.D. Dissertation (Universitat Autònoma de Barcelona, 2011).

[33]`http://www-personal.umich.edu/~bpl/larips/bachtemps.html`

[34]Figure source: `http://commons.wikimedia.org/wiki/File:Das_Wohltemperirte_Clavier_titlepage.jpg` (public-domain image).



The decoration is a series of loops of three different kinds: single loops, "double" loops (loops with an inner loop), and "triple" loops (with a "knot" of two overlapping loops inside the main loop). Not counting smaller flourishes on the ends, there are eleven loops, which suggests that these represent the consecutive intervals in the circle of fifths. In Lehman's interpretation, this decoration is best read with the page upside-down, as shown below.

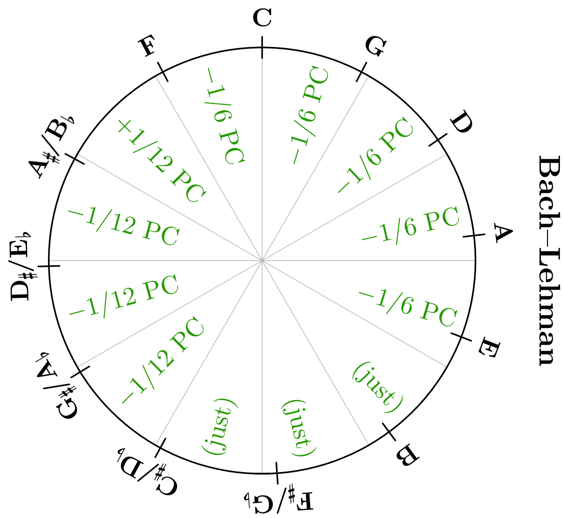


In this case, the five triple loops appear together on the left. These loops, being the most elaborate, are the intervals with the most tempering, and the relevant intervals are the central ones: F–C, C–G, G–D, D–A, and A–E. As we have seen in meantone temperaments and unequal closed temperaments, these intervals correspond to most-used keys, and benefit from a relatively strong tempering. Curiously, there is what looks like a "C" drawn in the manuscript in the "proper" position in the loop sequence. However, this is probably a serif for the C in *Clavier*,[35] although it is possible that it is pulling double-duty as a serif and an interval marker.

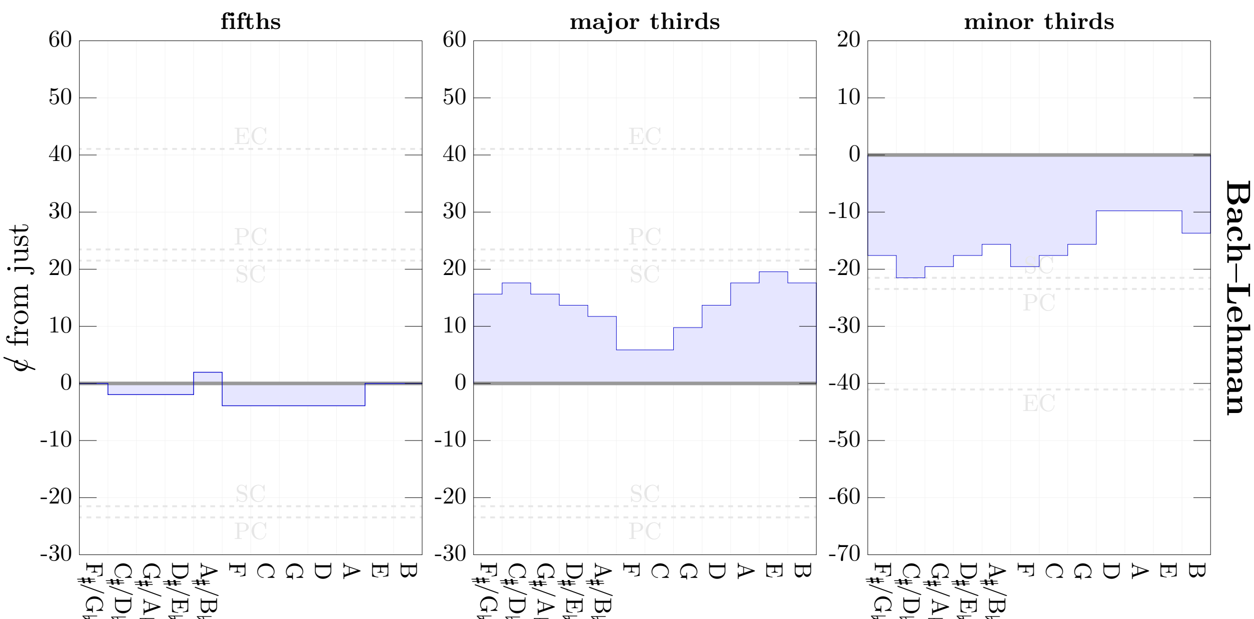
The next three single loops represent just intervals, and the final three double loops represent less-tempered intervals than the triple-loop intervals. The double-inner loop vs. the single-inner loop in triple and double loops, respectively, means the tempering in the weaker case is *half* that in the stronger case. Based in part on standard 18th-century practice, Lehman selects 1/6 of a Pythagorean comma for the stronger

[35]Mark Lindley and Ibo Ortgies, *op. cit.*, p. 615.

tempering, and thus 1/12 comma for the weaker tempering. The circle of fifths for the **Lehman–Bach temperament** below summarizes all of this.

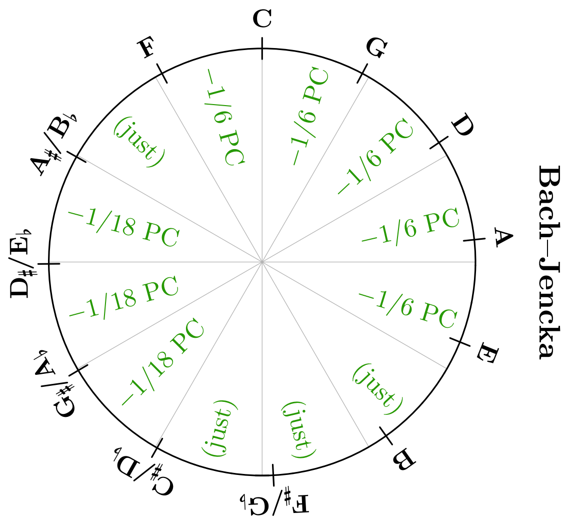


Note that the remaining interval, A♯–F, must be *sharp* by 1/12 of the Pythagorean comma for the temperament to work out to a total of 1 PC. The result is a temperament that is kind of like Vallotti, except that the 1/6 PC in the Vallotti E–B interval is distributed elsewhere in the raised keys, and of course the "wide" interval.

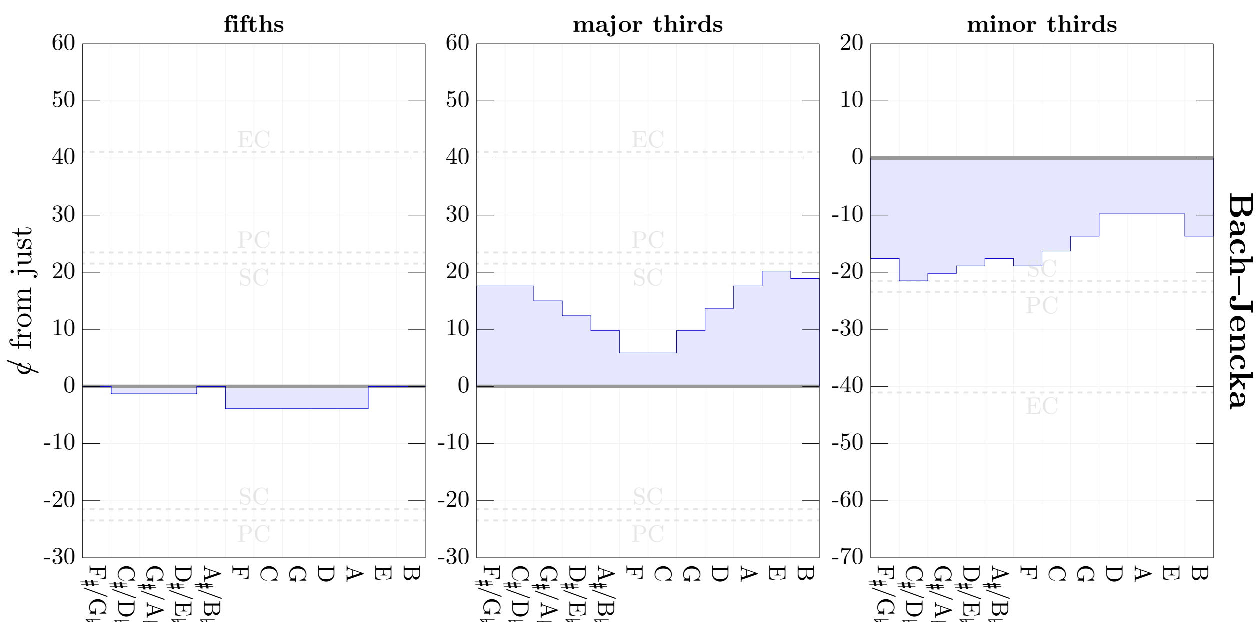


Daniel Jencka proposed an interesting variation on Lehman's approach.[36] The idea is to notice that the small flourish on the right-hand end of the loop decoration (in the inverted picture) looks like a (small) single loop. This suggests the "left-over" interval should in fact be pure. Jencka also proposes that the double and triple loops merely represent different temperaments, one not necessarily being double the other. In this case, to preserve the 1/6-PC temperament of the triple-loop intervals, this fixes the double-loop intervals to be tempered by 1/18 PC. This **Jencka–Bach temperament** is summarized in the circle of fifths below.

[36] Daniel Jencka, "The Tuning Script from Bach's Well Tempered Clavier: A Possible 1/18th PC Interpretation," available online at `https://web.archive.org/web/20120301105734/http://bachtuning.jencka.com/essay.htm` (2005, revised 2006). This essay reference by a letter in *Early Music* **33**, 545 (2005), `http://muse.jhu.edu/journals/emu/summary/v033/33.3correspondence.html`.



Again, this is somewhat Vallotti-like, with the E–B tuned just, and the 1/6 PC from this interval distributed equally over three other intervals.



To hear these in action, here is the Lehman–Bach temperament in triads,

(by fifths) (chromatic)

and here is the Jencka–Bach temperament in triads.

(by fifths) (chromatic)

As with the other unequal closed temperaments, all the triads sound more or less reasonble, some better than others.

5.7 Exercises

Problem 5.1

Show that the (quarter-comma) meantone wolf can be written as a pure fifth, less 1/4 of a syntonic comma, plus an enharmonic comma (diesis). (See Problems 2.2 and 4.1.)

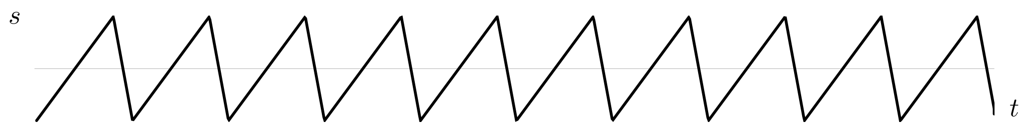
Problem 5.2

In Werckmeister III, is the minor-third interval (C–E♭) more or less pure than in Pythagorean tuning?

Chapter 6

Musical Examples

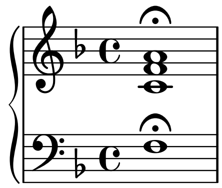
In this chapter, we will play a few short musical examples in the various temperaments, to get a better sense of how the temperaments "perform" in music. All the audio samples for a given music clip are synthesized from the same MIDI source. The synthesizer sound is a simple, asymmetric triangle wave, as in the plot below (showing the signal s in arbitrary unites as a function of time t).



This is the same waveform used for most of the sound samples in earlier chapters. This produces a sound somewhat like a harmonium (hand-pumped reed organ). This waveform makes the differences in temperament particularly easy to hear, because all harmonics are present, and the harmonics are relatively strong—thus differences in temperament show up as differences in the beating of various harmonics.

6.1 Example 1: Central Key

The first musical example is the first few bars from "*Fröhlicher Landmann* (*Album für die Jugend*, Op. 68, No. 10)," by Robert Schumann (1810-1856). Before getting on to the main song, listen to the first main chord in the song, shown below.



This is an F-major chord. Notice in the audio clips how this sounds particularly smooth or sweet in just tuning,

and in meantone tuning,

while it sounds more tense in Pythagorean tuning

and equal temperament.

Here is the chord in all temperaments for easy comparison:

Basic temperaments:

Pythagorean:

just:

equal temperament:

Meantone temperaments:

1/4-comma meantone:

1/3-comma meantone:

1/5-comma meantone:

1/6-comma meantone:

Unequal, closed temperaments:

Werckmeister III:

Kirnberger II:

Kirnberger III:

Neidhardt II:

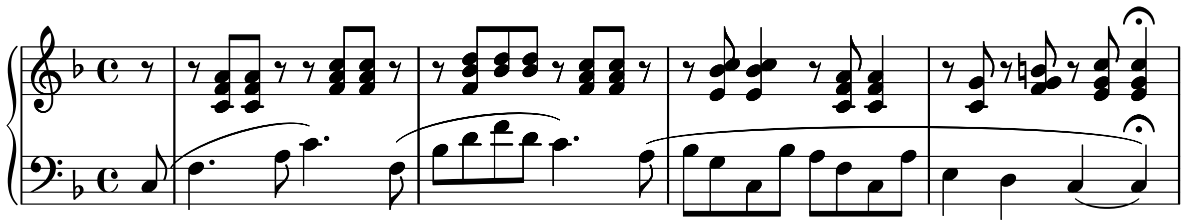
Vallotti:

Young II:

Lehman–Bach:

Jencka–Bach:

Now on to the full musical example, as shown below.



The major chords (for example, the first three chords and the last chord) are major chords, and again these all sound particularly nice in just tuning

and meantone temperament.

These are more tense in equal temperament,

and particularly Pythagorean tuning.

The meantone and just examples are especially nice, because of the greater contrast between the consonant and dissonant chords. For example, in the last measure, the second-last chord has a dissonant F–G pair that resolves to the C-major chord. The release of tension here is better when the final chord is more consonant.

Again, the excerpt is here in all the temperaments for easy comparison.

Basic temperaments:

Pythagorean:

just:

equal temperament:

Meantone temperaments:

1/4-comma meantone:

1/3-comma meantone:

1/5-comma meantone:

1/6-comma meantone:

Unequal, closed temperaments:

Werckmeister III:

Kirnberger II:

Kirnberger III:

Neidhardt II:

Vallotti:

Young II:

Lehman–Bach:

Jencka–Bach:

6.2 Example 2: Three Sharps

The next example is "Prélude (Op. 28, No. 7)" by Frédéric Chopin.

This piece is in a less-central key (G major). The main thing to notice is how the presence of a wolf interval causes serious problems. This is particularly true in meantone,

which you can compare, for example, to the neutral equal temperament:

Here is the Prélude in all temperaments:

Basic temperaments:

Pythagorean:

just:

equal temperament:

Meantone temperaments:

1/4-comma meantone:

1/3-comma meantone:

1/5-comma meantone:

1/6-comma meantone:

Unequal, closed temperaments:

Werckmeister III:

Kirnberger II:

Kirnberger III:

Neidhardt II:

Vallotti:

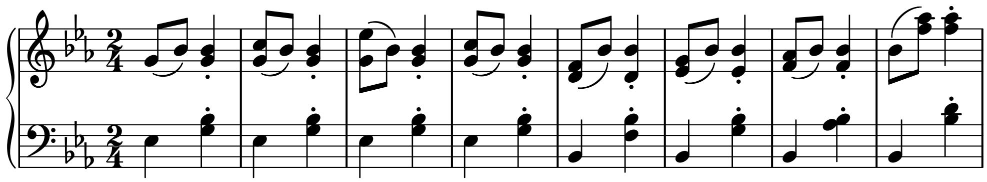
Young II:

Lehman–Bach:

Jencka–Bach:

6.3 Example 3: Three Flats

And now for something a bit more challenging. This is the introduction to the first of “Six Ecossaises” by Ludvig van Beethoven (1770–1827). The key is E♭ major, or three “notches” counterclockwise on the circle of fifths. Because of the allegro tempo and staccato notes, the differences in temperament are harder to hear.



But notice thatthe first chord is a G–B♭, or a minor third (the third and fifth from the E♭-major triad), so third-comma meantone actually shines in the first part of this excerpt.

However, the last measure contains an inverted A♭–F interval, which crosses the wolf interval, and sounds particularly *bad* in meantone, especially third-comma, but also in quarter-comma.

The last is something of a dissonant chord anyway (B♭ dominant 7, which contains a diminished fifth), because it leads into a repeat.

Basic temperaments:

Pythagorean:

just:

equal temperament:

Meantone temperaments:

1/4-comma meantone:

1/3-comma meantone:

1/5-comma meantone:

1/6-comma meantone:

Unequal, closed temperaments:

Werckmeister III:

Kirnberger II:

Kirnberger III:

Neidhardt II:

Vallotti:

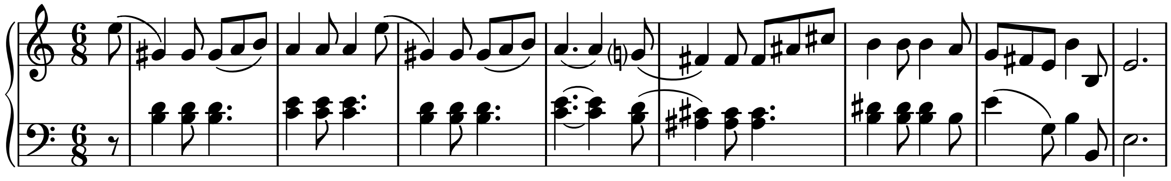
Young II:

Lehman–Bach:

Jencka–Bach:

6.4 Example 4: Diminished–Minor Cadence

We end with another Schumann composition, "*Sizilianisch* (*Album für die Jugend*, Op. 68, No. 11)."



The key is A minor (same signature as C major), and in the first two bars, moves from an inverted A♭ diminished 5 chord in the first measure (G♯–B–D, two stacked minor thirds) to an inverted A minor (A–C–E). The diminished 5 chord is *supposed* to be dissonant (so it resolves to the A minor), but in temperaments with a G♯–E♭ wolf, such as meantone, it doesn't sound very nice. (On the circle of fifths, the diminished 5 consists of the root, the minor third at 3 notches counterclockwise, and the diminished 5 at 6 notches clockwise, so it covers a large part of the circle.) Quarter-comma meantone,

and third-comma are good examples.

Note how the A minor sounds okay in quarter-comma meantone, but not so much in third-comma, because of the out-of-tune fifths in the latter case. The just-temperament example here is a good illustration of why it isn't generally useful. Pythagorean manages to do well on these two chords, because they don't cross the F♯–D♭ wolf interval.

In the fifth full measure, an F♯-major chord (F♯–C♯–A♯) moves to a B major third interval (B–D♯). Both of these manage to hit both the meantone and Pythagorean wolf intervals, and sound pretty badly. Equal temperament fares pretty well here.

Basic temperaments:

Pythagorean:

just:

equal temperament:

Meantone temperaments:

1/4-comma meantone:

1/3-comma meantone:

1/5-comma meantone:

1/6-comma meantone:

Unequal, closed temperaments:

Werckmeister III:

Kirnberger II:

Kirnberger III:

Neidhardt II:

Vallotti:

Young II:

Lehman–Bach:

Jencka–Bach:

Chapter 7

Helmholtz's Theory of Consonance and Dissonance

Now having thoroughly studied the concepts of intervals, we can revisit the concepts of consonance and dissonance, and be a bit more precise about their meaning and origin.

Recall that a dissonant interval is the golden ratio (about 1.61803 : 1), and here is the listening example again, with the interval played as a triangle wave:

Here is the same interval played as sine waves (pure tones); is it more consonant or dissonant? (Make sure not to listen to these clips at a high volume.)

As we will discuss, the first clip will tend to sound more dissonant, and while the second clip clearly sounds the same interval, the two tones don't "clash" as much as the first.

7.1 Dissonance and the Critical Band

Hermann Helmholtz developed the original model for mathematically describing the consonance or dissonance of intervals,[1] In more modern terms, we can describe consonance and dissonance with essentially the same theory, explained in terms of the critical band of the human ear. Recall that two tones played together, very close in pitch, will beat—they will sound like a simple tone, but with amplitude modulation occurring at the difference frequency. This is because the two pitches stimulate essentially the same area along the basilar membrane of the cochlea. If the two tones are very far apart in frequency, then they stimulate different regions along the basilar membrane, and you can perceive them as two distinct pitches. In between these extremes, there is a range of intervals where the two pitches will stimulate *different* but *overlapping* regions along the basilar membrane. The two pitches are not perceived as a single pitch or as two distinct pitches, and the resulting "confusion" in the ear leads to a "rough" quality to the sound. This roughness is the origin of dissonance.

To recall these different sound qualities, the following sound clip plays two sine waves together. One stays at middle C and the other starts at middle C and slowly sweeps (logarithmically) upward by one octave. Listen to the beats at first, transitioning to a rough sound as the beats speed up, and finally for the rest of the clip (and the majority of the duration of the clip), the two pitches sound distinctly.

[1] Hermann L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, second English edition translated by Alexander J. Ellis (Dover, 1954), corresponding to the fourth German edition (1877, with notes and additions to 1885) (ISBN: 0486607534).

So in terms of musical intervals, what is this range of the roughness? This is easy to try out. A minor second (semitone), with a middle C and the nearest C♯ in just intonation, sounds very rough:

A major second (whole tone), with a middle C and the nearest D in just intonation, also sounds rough:

By the time we get to a minor third (middle C and E♭ in just intonation), the sound is fairly smooth,

and a major third (middle C and E in just intonation) the roughness is clearly gone:

From the sweeping-pitch sound clip above, we know that larger intervals lack the rough quality. So from this, we can conclude that *small* intervals (semitones and whole tones) are *dissonant*, and *large* intervals are *consonant*.

7.2 Dissonance and Partials

However, that's not the *whole* story. As we already hinted in the golden-mean example above, consonance or dissonance also depends on the *tone quality* of the pitches involved. Most musical sounds contain harmonics as well as the fundamental pitch (i.e., the first harmonic), and any roughness caused by "clashes" (mismatches in frequencies) of harmonics can also cause dissonance. So while the major seventh (here, middle C and the next B up, in just intonation), when played as two sine waves, lacks roughness,

the same interval played in triangle waves sounds dissonant:

This is because the first harmonic of the B and the second harmonic of the C form together a minor second interval, which is dissonant, and gives the musical "tension" of this interval. The minor seventh is similarly dissonant, with the same two harmonics forming a major second.

The tritone (C to F♯) is another dissonant interval that is important in music theory, as it causes tension in dominant seventh chords that drives chord progressions forward. In some sense, the most "natural" tuning of the tritone is the equal-temperament tuning of a $2^{6/12} = \sqrt{2}$ ratio, because the tritone is its own inversion (and thus two tritones should combine to form an octave). This sounds dissonant, and we expect it to, because the ratio is an irrational number, and not especially close to any simple rational number.

But recall that in just intonation, the ratio is 7/5, which again is fairly far from the equal-temperament tritone (almost a third of a semitone). We might expect this one to sound better, and it does:

However, it is not as consonant as, say, the perfect fifth despite begin a relatively simple, rational number—indeed, precisely *because* it is not as simple a ratio as the perfect fifth (3/2). We can understand this as follows: the second harmonic of the F♯ is a ratio of $2 \times (7/5) = 2.8$, which is close to the third harmonic of the C. In fact the interval between 2.8 and 3 is $3/2.8 = 1.071$, which is between a minor and a major second, so we expect a clash to occur here. However, this clash is only in the second and third harmonics, which tend to be weaker than the fundamentals, and so we don't expect a dissonance like the minor second (clash between first harmonics) or the major seventh (clash between a first and a second harmonic). In fact, the reasonably consonant sound here from the first harmonics tends to mask this clash (and other clashes) of the harmonics.

To analyze the consonance or dissonance of an interval, in general we have to consider every harmonic of each pitch, and see whether there is any near-coincidence with any other harmonic (but, of course, not near enough to be a good match). Searching for clashes by hand is a bit cumbersome, but can of course be done fairly easily by computer. In practice, we only really need to consider the first few harmonics, as higher harmonics rapidly become small in amplitude for typical musical sounds. And even for the more important harmonics, as we have already seen, clashes in the harmonics may be of varying importance, depending on the tone quality of the pitches, which determine how important the clashing harmonics are to the overall sound. The main point to understand here is that consonance (and thus dissonance) is not a black or white phenomenon, but rather more like a continuum of grey. On one extreme, unisons or octaves have harmonics that match perfectly, and are thus highly consonant. The other extreme is characterized by small intervals, where the clashing fundamentals produce the rough, dissonant quality in the ear. Other intervals fall somewhere in between, due to clashing partials; the higher the partials involved in the clash, the smaller the tendency towards dissonance, because the high partials tend to be weak.

To understand the more consonant intervals, though, is relatively easy. For example, the octave interval as we mentioned, is highly consonant, because the octave is a harmonic of the lower note. Therefore, every harmonic of the higher pitch is also a harmonic of the lower pitch, and there can't be any clashes. The perfect fifth (3/2) seems a bit more complicated, because it is not a harmonic of the lower pitch. However, both pitches are harmonics (the second and third, respectively) of the same pitch, one octave below the lower pitch. Thus, their harmonics fall on the same harmonic series of a single pitch, so they avoid near clashes. The same argument can be applied to other intervals like the major third (5/4), just tritone (7/5), and so on. However, as the numbers in the ratio increase, the two pitches become increasingly higher harmonics of a progressively lower "missing fundamental" (fifth and seventh harmonics in the case of a tritone, and for a dissonant interval of 51/50, the 50th and 51st harmonics). The subtlety in this argument is that as the harmonic orders become large, clashes become impossible to rule out, because the high harmonics of a very low pitch are closely spaced together, and thus neighboring harmonics can clash. It is only for simple ratios that we expect the harmonics to be spaced well apart, and thus to avoid any clashes. With this in mind, however, we can similarly analyze more complicated combinations of pitches (chords). For example, the C major triad (C, E, and G, combining the major third and perfect fifth), has a consonant sound (in just intonation here):

The two intervals are 5/4 for the major third and 3/2 for the perfect fifth. Notice that multiplying both of these ratios by 4 gives whole numbers (5 and 6, respectively), so the three pitches of the major triad are in the ratio 4 : 5 : 6. Again, these are part of the same harmonic series, and not too high in the series, so consonance is exactly what we expect.

7.3 Consonance and Dissonance in Different Registers

We have so far talked about the consonance or dissonance of pitch intervals, and the critical-band model for explaining dissonance. One important feature of the critical band is that it depends on frequency. Thus, we expect this frequency dependence to be reflected in the consonance or dissonance of intervals in different registers (frequency ranges).[2] To summarize the frequency dependence of the critical band, we can use the model[3]

$$\Delta f_{\mathrm{CB}} = 24.7\,(0.00437 f + 1\ \mathrm{Hz}), \tag{7.1}$$

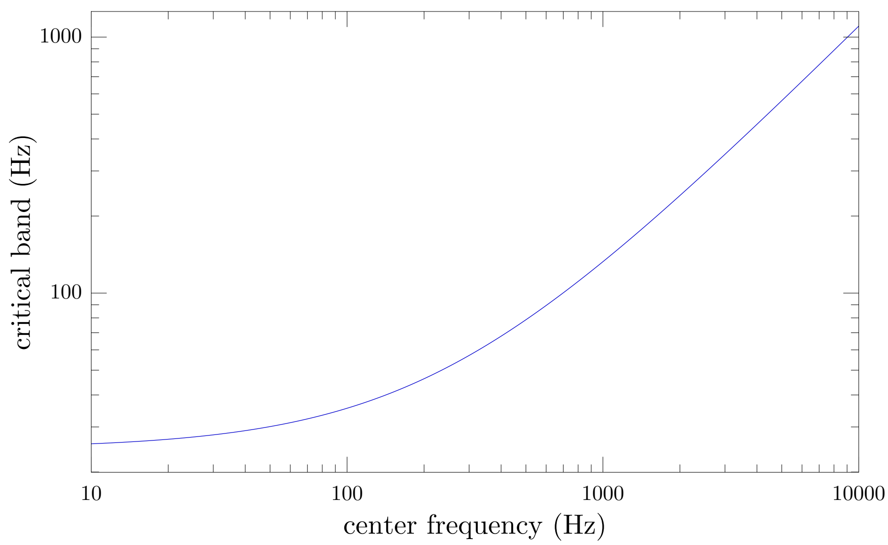
[2]For an extensive discussion of measurements and comparisons of consonance intervals and critical bands, see Donald D. Greenwood, "Critical bandwidth and consonance in relation to cochlear frequency-position coordinates," *Hearing Research* **54**, 164 (1991) (doi: 10.1016/0378-5955(91)90117-R).

[3]This is the "equivalent rectangular bandwidth" (ERB) model of Brian R. Glasberg and Brian C. J. Moore, "Derivation of auditory filter shapes from notched-noise data," *Hearing Research* **47**, 103 (1990), Eq.(3) (doi: 10.1016/0378-5955(90)90170-T).

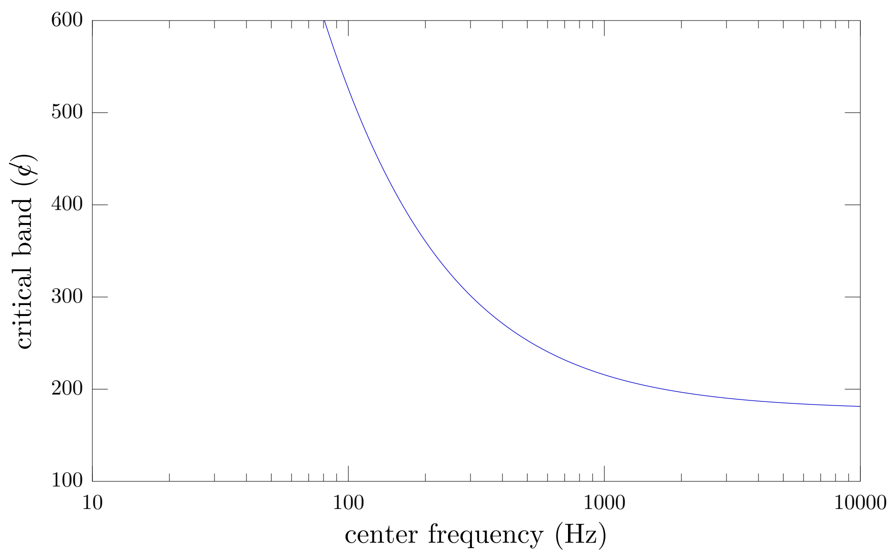
where f is the center frequency (in Hz). This is a measure of the *difference* between two frequencies f and $f + \Delta f_{\mathrm{CB}}$ corresponding to the critical band. To express this as a *ratio* between the same two frequencies, we instead want

$$\frac{f + \Delta f_{\mathrm{CB}}}{f} = 1 + \frac{\Delta f_{\mathrm{CB}}}{f} = 1.1079 + \frac{24.7\ \mathrm{Hz}}{f}. \tag{7.2}$$

This interval *decreases* as frequency increases, and ranges from a frequency ratio of 1.21 (325 ¢, or about a minor third) at 250 Hz to 1.12 (197 ¢, or about a major second) at 2 kHz. The intervals of maximum dissonance are somewhat smaller than the critical band (in the range of 30–40%), or specifically about 20 Hz at the 250 Hz center frequency (133 ¢) and 70 Hz at the 2 kHz center frequency (60 ¢).[4] The critical band model (7.1) is plotted below,



and here is the same model plotted as an interval as in Eq. (7.2), but converted to ¢:



The point of all this is as follows: The critical band, as a musical interval, gets *smaller* at higher frequencies. Thus, a dissonant musical interval at *low* frequencies will tend to be *less* dissonant at *high* frequencies. This is because of the smaller critical band at higher frequencies, so that the musical interval will tend to move outside the dissonance band.

[4] Measurements from those quoted in Fig. 5, Greenwood, *op. cit.*

Chapter 8

Continued Fractions: Why a 12-Note Scale?

In Pythagoras' construction of the musical scale, we saw that by stacking fifths, you don't *quite* ever get back to the original note. In that case, why is a choice of 12 notes reasonable for the scale, if the circle of fifths doesn't close? Is there some better choice for the number of fifths, either smaller or larger? Equivalently, we have looked at the equal-temperament (12-EDO) scale and a couple of other EDO's. Is there a choice of N-EDO that gives better intervals than $N = 12$? We've already hinted at the answers, but here we'll look at the math that lets you deduce the answer.[1] Although continued fractions and irrational numbers were familiar to the ancient Greeks, and they knew of the 53-division scale that we will come to,[2] it is not certain they thought this way. The continued-fraction method that we are going through here is attributed to Drobisch in 1855.[3]

8.1 Continued Fractions

The mathematical tool that we will need here is a **continued fraction**. Suppose we have some number A. We can write this as a continued fraction, for example, as

$$A = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}}, \tag{8.1}$$

where $a_0, \ldots, a_4$ are numbers—specifically integers, or whole numbers. More specifically, a_0 could be positive or negative (or zero), but $a_1, \ldots, a_4$ are positive integers. What we've written here is just a particular example with four "layers" in the fraction. Depending on A, we may need more or fewer layers (i.e., we may need to include $a_5, a_6, \ldots$).

The continued fraction is nothing more than a bunch of nested fractions. It's best to try this out with an example. Suppose that the numbers $a_0, \ldots, a_4$ are the numbers $1, \ldots, 5$, so that

$$A = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}}. \tag{8.2}$$

[1] The basic idea that we are exploring in this appendix is nicely described by Manfred Schroeder, *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise* (W. H. Freeman, 1991), pp. 99-101 (ISBN: 0716721368). See also J. M. Barbour, "Music and Ternary Continued Fractions," *The American Mathematical Monthly* **55**, 545 (1948) (doi: 10.2307/2304456).

[2] J. Murray Barbour, *Tuning and Temperament: A Historical Survey*, 2nd ed. (Michigan State College, 1953) (ISBN: 0486434060), p. 123. First edition available online at `https://archive.org/details/tuningtemperamen00barb`.

[3] M. W. Drobisch, "Über musikalische Tonbestimmung und Temperatur," *Abhandlungen der Mathematisch–Physischen Classe der Königlich Sächsischen Gesellschaft der Wissenschaften* **2**, 1 (1855). See description in J. M. Barbour, "Music and Ternary Continued Fractions," *The American Mathematical Monthly* **55**, 545 (1948) (doi: 10.2307/2304456).

Then we can just work out the value by starting at the innermost fraction (the one at the "tail end"), and sum the fractions to simplify it. We can do this thusly:

$$\begin{aligned} A &= 1+\frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{5}}}} \\ &= 1+\frac{1}{2+\frac{1}{3+\frac{1}{\left(\frac{21}{5}\right)}}} = 1+\frac{1}{2+\frac{1}{3+\frac{5}{21}}} \\ &= 1+\frac{1}{2+\frac{1}{\left(\frac{68}{21}\right)}} = 1+\frac{1}{2+\frac{21}{68}} \\ &= 1+\frac{1}{\left(\frac{157}{68}\right)} = 1+\frac{68}{157} = \frac{225}{157}. \end{aligned} \tag{8.3}$$

So this continued fraction is just another way to write out the fraction 225/157.

At this point, it is useful to know that

1. Any continued fraction with a finite number of "layers" or coefficients a_n represents a ratio of two integers (i.e., a **rational number**).
2. Any rational number can be written as a continued fraction, and the representation as a continued fraction is unique for every rational number. (An exception to this occurs if the last number in a continued fraction is 1, in which case there are two different continued fractions that represent the same number. See Problem 8.3.)

On the last point, again, different numbers of layers/coefficients a_n are necessary to represent different rational numbers.

8.1.1 Shorthand

Writing out nested fractions as we did above is a bit cumbersome. A common shorthand is to write out the continued fraction in Eq. (8.1) as

$$A = a_0+\frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\frac{1}{a_4}}}} = [a_0, a_1, a_2, a_3, a_4]. \tag{8.4}$$

(continued-fraction shorthand)

In other words, we can write the continued fraction in Eq. (8.3) as $[1, 2, 3, 4, 5] = 225/157$.

8.2 Rational Approximations of Irrational Numbers

So what's the point of all these continued fractions, that seem cumbersome and take up a lot of space on the page (at least, when you don't use the shorthand notation)? We'll come to the answer in just a bit, but first we need to talk about the relationship between rational and irrational numbers. Recall that an **irrational** number is one that *can't* be written as a rational number—a ratio of two integers. However, an irrational number can always be approximated *as well as you like* by rational numbers. You're familiar with this, even if you don't think so. Take π, for example:

$$\pi = 3.14159265\ldots \tag{8.5}$$

The ellipses at the end indicate that this is an irrational number by indicating that the decimal digits never end. The most intuitive rational approximants are just truncating to different decimal places:

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \quad \ldots \tag{8.6}$$

(We could get even more accurate approximants by *rounding* the results properly, but the truncation is more in the spirit of how we will handle continued fractions in a bit.) These are rational, because decimal numbers that truncate are just ratios, with power-of-ten denominators:

$$3, \quad \frac{31}{10}, \quad \frac{314}{100}, \quad \frac{3141}{1000}, \quad \frac{31\,415}{10\,000}, \quad \frac{314\,159}{100\,000}, \quad \ldots \tag{8.7}$$

Each fraction here is better (more accurate) than the one before it. The first has an error of the order of 10%, the next 1%, the next 0.1%, and so on.

8.2.1 Rational Approximations via Continued Fractions

Instead of using fractions based on decimal numbers to approximate irrational numbers, it turns out to be far better to use fractions based on continued fractions. Generally speaking, the idea is similar to that of the decimal representation, in that representing an *irrational* number A requires an unterminating continued fraction:

$$A = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} = [a_0, a_1, a_2, a_3, a_4, \ldots]. \tag{8.8}$$

But we can always make up a rational approximation by truncating the continued fraction to, say, $[a_0, a_1, a_2]$.

To find *any* continued fraction, first look at Eq. (8.8). Note that a_0 is the integer part of A, and the rest is the fractional part. So it's easy to find a_0. Then, we can take away this fractional part,

$$A - a_0 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} = \frac{1}{[a_1, a_2, a_3, a_4, \ldots]}, \tag{8.9}$$

and then invert both sides:

$$\frac{1}{A - a_0} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}} = [a_1, a_2, a_3, a_4, \ldots]. \tag{8.10}$$

The result is another continued fraction starting with a_1, and a_1 is the integer part of what remains. So you just keep doing this, finding and removing the integer part and inverting the result, to find all the coefficients.

8.2.2 π

To get a feel for this, let's try this out for π. The integer part is 3, so $a_0 = 3$, and [3] is the simplest continued-fraction approximation for π. Then remove this and invert:

$$\frac{1}{\pi - 3} = \frac{1}{0.141\,592\,65\ldots} = 7.062\,513\,3\ldots. \tag{8.11}$$

Then a_1 is the integer part of this, or $a_1 = 7$. To continue,

- The remainder is $0.062\,513\,3\ldots$, which is $15.996\,594\,4\ldots$. Thus, $a_2 = 15$.
- Removing the 15 and inverting, we get $1/0.996\,594 = 1.003\,417$, so $a_3 = 1$.
- Removing the 1 and inverting, we get $1/0.003\,417\ldots = 292.635$, so $a_4 = 292$.

And so on. Thus we can write π as a continued fraction as

$$\pi = [3, 7, 15, 1, 292\ldots]. \tag{8.12}$$

(continued fraction for π)

We can truncate and round up to find the successively more accurate rational approximations

$$\pi \approx [3], \quad [3,7], \quad [3,7,15], \quad [3,7,15,1], \quad [3,7,15,1,292], \quad \ldots. \tag{8.13}$$

If we work out all these continued fractions (just like the example before for $[1, 2, 3, 4, 5]$), the fractions are

$$\pi \approx 3, \quad \frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}, \quad \frac{103\,993}{33\,102}, \quad \ldots. \qquad \text{(continued fraction approximations for } \pi) \quad (8.14)$$

Note that the fractions for $[3, 7, 16]$ and $[3, 7, 15, 1]$ are the same (we obtained the first one by rounding). We can also write this as

$$\pi \approx 3, \quad 3.142\,857\,143\ldots, \quad 3.141\,592\,920\ldots, \quad 3.141\,592\,920\ldots, \quad 3.141\,592\,654\ldots, \quad \ldots. \qquad (8.15)$$

in decimal notation to see how these fractions are approaching π. These approximants have fractional errors of 0.05, 0.0004, 3×10^{-5}, 8×10^{-8}, and 2×10^{-10}, respectively.

8.2.3 Accuracy

The reason that it's nice to use continued fractions to approximate irrational numbers is that they are *the most accurate approximants*. To be more specific, suppose we try to represent some irrational number A by a fraction r/s. If we got this fraction by truncating/rounding the continued fraction, then it turns out to be the best choice in the sense that there is no better fraction with a denominator less than or equal to s.

Also remember that in the decimal-based approximants, the error was typically of the order $1/s$. It turns out the error in a continued-fraction-based approximant is more like $1/s^2$, which is much smaller, especially as s gets larger for more accurate approximants.

8.3 Rational Approximations of the Octave by Fifths

Now back to the original question of why twelve notes in the scale? Back in Section 2.5.1, we argued that there was a fundamental problem in stacking just perfect fifths (3/2 ratio) to obtain a whole number of octaves. That is, if we take n just fifths and set them equal to m octaves,

$$\left(\frac{3}{2}\right)^n = 2^m, \qquad (8.16)$$

(defining relation for stacking fifths)

the problem is that we can never find integer values for n and m that satisfy this equation—the problem is that 3 and 2 are relatively prime, so that no power of 3 can ever match any power of 2.

8.3.1 Restatement of the Problem

But let's proceed anyway, and see how well we can do. If we multiply through by 2^n,

$$3^n = 2^{n+m}, \qquad (8.17)$$

and then taking the logarithm of the equation,

$$n \log 3 = (n + m) \log 2, \qquad (8.18)$$

or

$$\frac{n + m}{n} = \frac{\log 3}{\log 2} \approx 1.584\,962\ldots. \qquad (8.19)$$

It will be a bit more convenient to simplify the fraction a bit. Using $(n + m)/n = 1 + m/n$, we can subtract 1 everywhere to obtain We can also subtract 1 from every part of this equation, with the result

$$\frac{m}{n} = \left(\frac{\log 3}{\log 2} - 1\right) \approx 0.584\,962\ldots. \qquad (8.20)$$

(setup for rational approximation)

Now clearly $(\log 3)/(\log 2) - 1$ has some value, so there is *some* solution for m/n. It's just that it's an *irrational* solution, which doesn't *directly* do us much good (we can't have a *fraction* of a note in a scale, except of course as some kind of comma).

8.3.2 Continued-Fraction Solution

However, what we *can* do is look for some good *rational approximations* to $(\log 3)/(\log 2) - 1$, and use *that* as the basis for our scales. And, of course, the way to do this is via continued fractions. Using the same procedure that we use for π, we have $a_0 = 0$, with remainder $0.584\,963\ldots$.

- Then $1/0.584\,963\ldots = 1.709\,511\ldots$, so $a_1 = 1$.
- Then $1/0.709\,511\ldots = 1.409\,420\ldots$, so $a_2 = 1$.
- Then $1/0.409\,421\ldots = 2.442\,474\ldots$, so $a_3 = 2$.
- Then $1/0.442\,474\ldots = 2.260\,016\ldots$, so $a_4 = 2$.
- Then $1/0.260\,016\ldots = 3.845\,906\ldots$, so $a_5 = 3$.
- Then $1/0.845\,906\ldots = 1.182\,164\ldots$, so $a_6 = 1$.
- Then $1/0.182\,164\ldots = 5.489\,547\ldots$, so $a_7 = 5$.
- Then $1/0.489\,547\ldots = 2.042\,704\ldots$, so $a_8 = 2$.

That is, we have

$$\left(\frac{\log 3}{\log 2} - 1\right) = [0, 1, 1, 2, 2, 3, 1, 5, 2, \ldots]. \tag{8.21}$$

This is the *exact, irrational* solution to Eq. (8.20).

8.3.2.1 Approximants and Scales

Then from Eq. (8.19), the idea is to use truncated continued fractions to set a *rational* value for the ratio m/n. This allows us to close the circle of fifths as well as possible, given a certain number of fifths (or notes in the octave).

The approximants are then

$$\left(\frac{\log 3}{\log 2} - 1\right) \approx [0], \quad [0,1], \quad [0,1,1], \quad [0,1,1,2], \quad [0,1,1,2,2], \quad [0,1,1,2,2,3], \quad [0,1,1,2,2,3,1], \quad [0,1,1,2,2,3,1,5], \quad [0,1,1,2,2,3,1,5,2], \quad \ldots \tag{8.22}$$

The corresponding ratios are

$$\left(\frac{\log 3}{\log 2} - 1\right) \approx 0, \quad 1, \quad \frac{1}{2}, \quad \frac{3}{5}, \quad \frac{7}{12}, \quad \frac{24}{41}, \quad \frac{31}{53}, \quad \frac{179}{306}, \quad \frac{389}{665}, \quad \ldots, \qquad \text{(scale approximants)} \tag{8.23}$$

each one a more accurate approximation to the exact, irrational value.

8.3.2.2 Interpretation

Now, how to interpret these? Well, remember that we can set m/n to any of these ratios, and these are supposed to be a "good" approximation to $(\log 3/\log 2 - 1)$. And in doing so, from Eq. (8.16), n is the number of stacked **just** perfect fifths, and m is the (nearly) equivalent number of octaves.

First, let's focus on the familiar case of 7/12. This says, that 12 stacked pure fifths (12 notes per octave) is very nearly 7 octaves. We already knew this: the Pythagorean comma is reasonably small. However, since this came from a continued fraction, we get another nice result, which is that it's not possible to do any better by using any *fewer* number of notes (fifths).

In fact, the continued fraction gives us a few options with fewer stacked fifths. The next-simple choice is 5 fifths make 3 octaves, but this already isn't very accurate: a stack of five pure fifths brings you to the major seventh, which is close to, but noticeably different from, the octave (i.e., it is a semitone away). This does, however, get you almost all of the diatonic major scale—you just need to add the perfect fourth. The simpler options are of course worse: three pure fifths gets you to the major sixth, and two fifths gets you a major second.

Thus, a stack of twelve fifths is the simplest choice that gets you reasonably close to returning to the original note ("reasonably" of course being a subjective term). Of course, we can now *add* extra notes if we want to construct a scale with a *smaller* analogue to the Pythagorean comma. However, the next scale that is an improvement on Pythagoras' choice is a 41-note scale, which is fairly unwieldy. To improve on that, we have to go to 53, 306 and then 665 stacked fifths! So Pythagoras was actually quite clever, in balancing good accuracy (a small Pythagorean comma) with a reasonable quantity of notes.

8.3.3 Application to EDO

This analysis also answers a slightly *different* question. Suppose that we consider various n-EDO scales. Which ones have "nice" fifths? That is, suppose that the fifth is the mth note in the scale. In n-EDO, the interval between notes is $2^{1/n}$. Thus, the interval for the fifth is $2^{m/n}$, and so we seek solutions to the equation

$$2^{m/n} = \frac{3}{2}. \tag{8.24}$$

(defining relation for "good fifth" EDO)

That is, we want our n-EDO to peg the just perfect fifth as well as we can. Taking the logs of both sides,

$$\frac{m}{n}\log 2 = \log\frac{3}{2}, \tag{8.25}$$

and if we divide through by $\log 2$, we get

$$\frac{m}{n} = \frac{\log 3}{\log 2} - 1. \tag{8.26}$$

This is exactly the same as Eq. (8.20), so we have already solved this problem.

This means that the ratios in Eq. (8.23) also give us directly the n-EDO interval corresponding to the fifth, and the denominators still give us the best choices for n to give nicely tuned fifths. The familiar example is 12-EDO, for which the fifth is

$$2^{7/12} = 1.498\ 307\ldots \approx \left(\frac{3}{2}\right)(-1.96\,¢)\,. \tag{8.27}$$

That is, the 12-EDO fifth is 1.96 ¢ flat of the just fifth.

The next best choice is 41-EDO, where the fifth is the 24th note in the scale. The ratio is

$$2^{24/41} = 1.500\ 419\ldots \approx \left(\frac{3}{2}\right)(+0.48\,¢)\,, \tag{8.28}$$

which is some improvement over 12-EDO (about 1/4 the comma for more than three times the notes). The next choice is 53-EDO, where the fifth is the 31st note in the scale; the ratio here is

$$2^{31/53} = 1.499\ 940\ldots \approx \left(\frac{3}{2}\right)(-0.068\,¢)\,, \tag{8.29}$$

which is considerably closer to the just fifth. The next choices are

$$2^{179/306} = 1.500\ 005\ 01\ldots \approx \left(\frac{3}{2}\right)(+0.0058\,¢)\,, \tag{8.30}$$

and

$$2^{389/665} = 1.499\ 999\ 901\ 5\ldots \approx \left(\frac{3}{2}\right)(-0.00011\,¢)\,, \tag{8.31}$$

which is indeed very close to the just fifth.

8.4 Exercises

Problem 8.1

(a) Work out the first 5 numbers in the continued-fraction representation of $2^{1/12}$.

(b) Use your answer in (a) to come up with a fraction that approximates $2^{1/12}$. How accurate is this approximant (in percent)?

Problem 8.2

What is the simplest (beyond "1") continued-fraction approximation to the cent? Compute this rational approximation in cents.

Problem 8.3

In certain cases, continued fractions aren't unique. The main problem comes about in writing down continued fractions for rational numbers, so the continued fraction truncates. For every rational number, there is a continued fraction that ends with one, and another that doesn't. As an example of this:

(a) Work out the value of the continued fraction $[0, 1, 2, 3, 1]$.

(b) Can you find an alternate continued fraction for the same number? Look at the first step of your work in (a).

Problem 8.4

Because frequency is inversely proportional to length of an ideal string, the frets of stringed instruments (like guitars) in equal temperament should have frets that effectively make the strings shorter by a factor of $2^{-1/12}$ for each successive fret. This is only an approximation, because it ignores tension changes due to the sideways displacement and the applied force involved in fretting, as well as effects due to the stiffness of the string.

This $2^{-1/12}$ is a bit hard to measure, so stringed-instrument makers use the **rule of 18** to place frets. This says that each string should "remove" 1/18th of the string's length.

Show how the rule of 18 arises from a continued-fraction approximation of $2^{-1/12}$.

Problem 8.5(!)

Use the procedure for working out continued fractions to argue that the golden ratio, defined as

$$\phi = \frac{1+\sqrt{5}}{2}, \tag{8.32}$$

corresponds to the continued fraction $[1, 1, 1, 1, \dots]$ (i.e. an infinitely continued fraction with all ones).

Hint: you should use the fact that the inverse of the golden ratio can be written

$$\frac{1}{\phi} = \frac{\sqrt{5}-1}{2} = \phi - 1. \tag{8.33}$$

You should prove this too!

Problem 8.6(!)

We argued that the best choices for N-EDO (in order to have a perfect fifth close to just intonation) are $N = 12, 41, 53, 306, 665, \dots$

(a) Fill in the next two numbers in that sequence, using the same reasoning.

(b) Compute the difference between the N-EDO fifth and the just fifth for these next two scales. Express your answers in cents.

Problem 8.7(!)

We set up this whole continued-fraction approach to choosing the number of notes in a scale by considering how well a stack of fifths matches an octave. Suppose instead that we are more concerned with how well a stack of *major thirds* matches an octave.

(a) Set up this problem in the same way as for the stack of fifths, and find at least the first 5 numbers in the continued-fraction representation of the irrational number you find.

(b) Obtain at least the first four rational approximants by truncating the continued fraction. What are the N-EDO scales that have the closest-to-pure major thirds?

Note that you can think of such N-EDO scales as having been tempered to remove the diesis (Problem 2.2).

Index